

ORDERING THE SPACE OF FINITELY GENERATED GROUPS

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ABSTRACT. We consider the oriented graph whose vertices are isomorphism classes of finitely generated groups, with an edge from G to H if, for some generating set T in H and some sequence of generating sets S_i in G , the marked balls of radius i in (G, S_i) and (H, T) coincide.

Given a nilpotent group G , we characterize its connected component in this graph: if that connected component contains at least one torsion-free group, then it consists of those groups which generate the same variety of groups as G .

The arrows in the graph define a preorder on the set of isomorphism classes of finitely generated groups. We show that a partial order can be imbedded in this preorder if and only if it is realizable by subsets of a countable set under inclusion.

We show that every countable group imbeds in a group of non-uniform exponential growth. In particular, there exist groups of non-uniform exponential growth that are not residually of subexponential growth and do not admit a uniform imbedding into Hilbert space.

1. INTRODUCTION

Our aim, in this paper, is to relate the following preorder on the set of isomorphism classes of finitely generated groups with asymptotic and algebraic properties of groups.

Definition 1.1. Let G, H be finitely generated groups. We write $G \rightsquigarrow H$, and say that G *preforms* H , if the following holds. There exist a finite generating set T of H and a sequence of finite generating sets S_1, S_2, \dots of G , with bijections $S_n \rightarrow T$ such that, for all $R \in \mathbb{N}$, if n is large enough then the balls of radius R in the marked Cayley graphs of (G, S_n) and (H, T) are isomorphic.

We denote by $\mathcal{C}(G, S)$ the Cayley graph of the group G with respect to the generating set S . Its edges are marked with the generator they correspond to.

If G preforms H , then we also say that H is *performed by* G .

Definition 1.1 can be interpreted in terms of the *Chabauty-Grigorchuk topology*, also called the *Cayley topology*, defined as follows. The *space of marked groups* is the set \mathcal{G} of pairs (G, S) with G a finitely generated group and S a finite ordered generating set, considered up to group isomorphism preserving the generating set. This is equipped with a natural topology, two marked groups (G, S) and (G', S') being close to each other if marked balls of large radius in the Cayley graphs $\mathcal{C}(G, S)$ and $\mathcal{C}(G', S')$ are isomorphic.

Chabauty considered this topological space in [18, §3]; he used it to describe the space of lattices in locally compact groups. Gromov [28, pages 71–72] used it to derive an effective version of his theorem on groups of polynomial growth. Grigorchuk [27] was the first to study this topology systematically; in particular, he used it to construct groups of

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24 wildly-oscillating intermediate growth, by approximating them in \mathcal{G} by solvable groups.
 25 For generalities on the the space of marked groups, see [19].

26 Definition 1.1 may then be formulated as follows: $G \preccurlyeq H$ if and only if the closure of
 27 the isomorphism class of G in the Chabauty-Grigorchuk topology contains H .

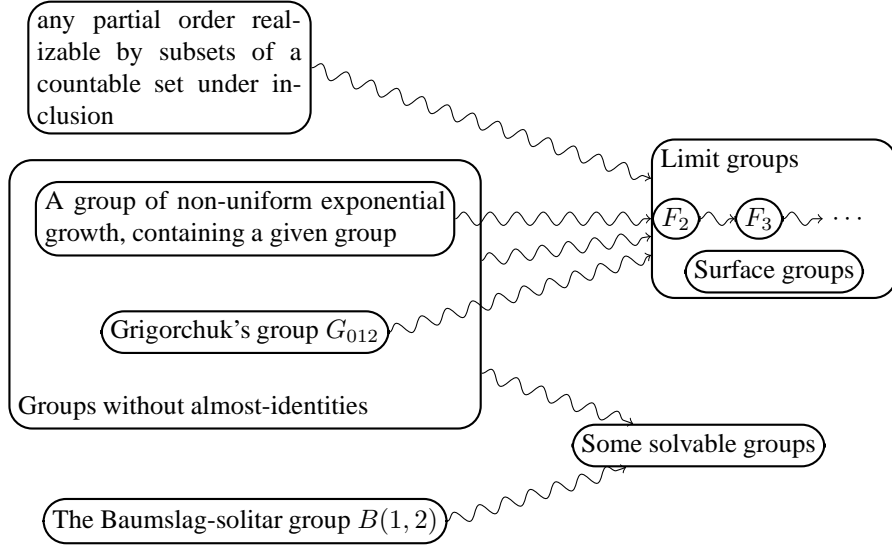
28 It is essential for our definition that we consider limits in the space of marked groups
 29 of a fixed group, letting only its generating set vary. Various authors have already con-
 30 sidered limits in the space of marked groups, not necessarily restricting to limits within
 31 one isomorphism class. Limits of one fixed group have been studied when this group is
 32 free: they coincide with limits groups, as shown by Champetier and Guirardel [19, Theo-
 33 rem 1.1]; see §6.1 for more references. Zarzycki [61] considers groups that are preformed
 34 by Thompson’s group F , and gives some necessary conditions for HNN extensions to ap-
 35 pear in this manner; Guyot [31, 32] considers groups that are preformed by G for some
 36 metabelian groups G , and identifies their closure in \mathcal{G} . On the other hand, groups that
 37 preform free groups are groups that have infinite girth for generating sets of fixed cardinal-
 38 ity. Olshansky and Sapir characterize them in [45] as groups without almost-identities, see
 39 also §6.2.

40 We recall that a *preorder* is a binary relation \preccurlyeq such that $A \preccurlyeq C$ whenever $A \preccurlyeq B$
 41 and $B \preccurlyeq C$ and such that $A \preccurlyeq A$ for all A . If furthermore ‘ $A \preccurlyeq B$ and $B \preccurlyeq A$ ’ imply
 42 $A = B$, then it is an *order*. A preorder is *directed* if every finite subset has an upper bound.
 43 It is easy to see that the relation ‘ \preccurlyeq ’ is a preorder, and that $G \preccurlyeq H$ does not depend on
 44 the choice of a finite generating set in H (see Lemmas 2.2 and 2.1 in the next section). It
 45 is also not difficult to see that the restriction of this relation to some classes of groups is
 46 an order; this happens, for example, for residually finite finitely presented groups, such as
 47 polycyclic groups (see Corollary 2.7). For some other classes of groups this is not true: for
 48 example, there exist solvable groups G admitting a continuum of non-isomorphic solvable
 49 groups which are equivalent to G under our preorder, that is, which both preform and are
 50 preformed by G . Nekrashevych gave in [43] examples of groups acting on rooted trees
 51 which are equivalent under our preorder.

52 In many cases, if A preforms B , then A “looks smaller” than B . Simple examples
 53 of this kind include: $\mathbb{Z}^m \preccurlyeq \mathbb{Z}^n$ if and only if $m \leq n$; free groups satisfy $\mathbb{F}_m \preccurlyeq \mathbb{F}_n$
 54 if and only if $m \leq n$; and the n -generated free groups \mathbb{V}_n in the variety generated by a
 55 torsion-free nilpotent group of nilpotency class c satisfy, for $m, n \geq c$, the same relation
 56 $\mathbb{V}_m \preccurlyeq \mathbb{V}_n$ if and only if $m \leq n$, see Theorem A. On the other hand, it may happen for A
 57 that preform B that the growth of A is larger than the growth of B ; we consider this in
 58 more detail in §1.3.

59 **1.1. The structure of components.** We view \preccurlyeq as specifying the edge set of an oriented
 60 graph with vertex set the isomorphism classes of finitely generated groups. In studying
 61 this graph, we may consider independently the *connected components* of its underlying
 62 unoriented graph. What do they look like? Which components admit an initial vertex? a
 63 terminal vertex? Given a connected component, does it have an upper bound? What is
 64 the group of preorder preserving bijections of a given component? Which groups’ *strongly*
 65 *connected component* are reduced to points, or have the cardinality of the continuum?

66 Unlike some other natural preorders, such as “being a subgroup”, “being a quotient
 67 group”, or “being larger” in the sence of Pride ($G \succcurlyeq_p H$ if H_1 is a quotient of G_1 , for
 68 respective quotients G_1, H_1 of finite-index subgroups of G, H by finite normal subgroups,
 69 see [49, 54]), the preorder that we consider in this paper has infinitely many connected com-
 70 ponents. An easy example is the connected component of \mathbb{Z} : it contains all infinite abelian

FIGURE 1. Some classes of groups and their relationship under \preceq

71 groups, and we describe the group of the order preserving bijections of this component in
 72 Proposition 3.7.

73 For a nilpotent torsion-free G group, its connected component is closely related to
 74 groups that generated the same variety as G .

75 **Theorem A** (= Proposition 4.6). *Let G be a finitely generated nilpotent group such that G
 76 and $G/\text{Torsion}(G)$ generate the same variety (i.e. satisfy the same identities). Then, for
 77 all $k \in \mathbb{N}$ large enough, G preforms the relatively free group of rank k in that variety.*

78 In particular, every finite set of such nilpotent groups has a supremum with respect to
 79 our preorder. We believe, in fact, that this last statement holds for all virtually nilpotent
 80 groups. However, if a nilpotent group G is not torsion-free, the connected component of G
 81 can be much smaller than the set of (isomorphism classes) of groups generating the same
 82 variety as G , see Corollary 4.12.

83 We show, on the other hand, that the preorder types that can occur are quite general,
 84 even within solvable groups of class 3, or within groups that preform free groups:

85 **Theorem B** (= Corollary 5.2 and Remark 6.9). *Let (X, \preceq) be a preorder. Then $(\mathcal{G}/\cong, \preceq)$
 86 contains (X, \preceq) as a subpreorder if and only if X has cardinality at most the continuum,
 87 and all the partial orders it contains are imbeddable in the partial order of subsets of \mathcal{B}
 88 under inclusion, for a countable set \mathcal{B} .*

89 Furthermore, the imbedding of (X, \preceq) can then be chosen to be within the set of iso-
 90 morphism classes of solvable groups of solubility class 3 or, alternatively, within the set of
 91 isomorphism classes of groups that preform \mathbb{F}_3 .

92 Thomas studies in [54] the complexity, with respect to the Borelian structure on \mathcal{G} , of
 93 Pride's "largeness" preorder and of the "being a quotient" preorder. He shows that these
 94 preorders are high in the Borel hierarchy (namely, \mathbf{K}_σ -universal). The preorder \preceq differs
 95 from the above mentioned preorders even if we forget the underlying Borelian structure:
 96 the quotients and largeness preorders have chains with cardinality the continuum, while
 97 (by Theorem B) chains for \preceq are countable.

1.2. **Groups larger or smaller than a given group.** Given a group G , how many groups preform G ? How many groups are preformed by G ? How big is the connected component of G ? What is its diameter?

We note that, if a group G is virtually nilpotent, then its component is countable. The number of groups that are preformed by G is countably infinite.

If G is a free group, a surface group, or more generally a non-abelian limit group (see §6.1), then there are countably many groups that are preformed by G , see [37, 52]. However, the connected component of G has the cardinality of the continuum, see Example 6.3.

We study the groups that preform free groups. Schleimer considered groups of unbounded girth (there are generating sets such that the smaller cycle in the Cayley graph is arbitrarily long) in an unpublished note [51], and they are intimately connected to groups that preform free groups, see Question 8.5. The latter are groups that do not satisfy an *almost-identity* [45]: a word whose evaluation vanishes on every generating set. Olshanskii and Sapir show in [45] that there are groups with non-trivial quasi-identities among groups satisfying no non-trivial identity.

In §6.3, we modify a criterion by Abért [1] about groups without identities to determine when a group has no almost-identity. This lets us answer negatively a question by Schleimer [51, Conjecture 6.2] that groups of unbounded girth have exponential word growth (see §1.3 for the definition of growth):

Theorem C (= Corollary 6.12). *The first Grigorchuk group G_{012} preforms \mathbb{F}_3 .*

Extending an argument by Akhmedov (see [4]), we give a criterion for a wreath product with infinite acting group to preform a free group:

Proposition D (= Proposition 6.15). *Let G and H be finitely generated groups, and suppose that H is infinite. Then the restricted wreath product $G \wr H := G^{(H)} \rtimes H$ preforms a free group if and only if at least one of the following conditions holds:*

- (1) G does not satisfy any identity;
- (2) H does not satisfy any almost-identity.

From this, we deduce (see Remark 6.21) that the connected component of the free group has diameter at least 3; this is in contrast with the nilpotent case, see Theorem A. There are solvable groups, and infinite free Burnside groups, at distance 2 from a free group.

See also subsection 2.4 where we discuss groups that preform a group containing a given subgroup.

1.3. **Growth of groups.** We finally give in §7 some new examples of groups of non-uniform exponential growth. Recall that, for a group G generated by a set S , its *growth function* counts the number $\nu_{G,S}(R)$ of group elements expressible as a product of at most R generators. The group has *exponential growth* if $\lambda_{G,S} := \lim_{R \rightarrow \infty} \sqrt[R]{\nu_{G,S}(R)} > 1$ and *subexponential growth* otherwise; it then has *polynomial growth* if $\nu_{G,S}$ is dominated by a polynomial, and *intermediate growth* otherwise. The existence of groups of intermediate growth was asked by Milnor in [42], and answered by Grigorchuk in [27], by means of his group G_{012} .

If G has exponential growth, then it has *uniform exponential growth* if furthermore $\inf_S \lambda_{G,S} > 1$. The existence of groups of non-uniform exponential growth was asked by Gromov in [29, Remarque 5.12]; see also [36]. The first examples were constructed by Wilson [59]; see also [8, 43, 60].

143 **Theorem E** (= Corollary 7.3). *Every countable group may be imbedded in a group G of*
 144 *non-uniform exponential growth.*

145 *Furthermore, let $\alpha \approx 0.7674$ be the positive root of $2^{3-3/\alpha} + 2^{2-2/\alpha} + 2^{1-1/\alpha} = 2$.*
 146 *Then G may be required to have the following property: there is a constant K such that,*
 147 *for any $R > 0$, there exists a generating set S of G with*

$$\nu_{G,S}(r) \leq \exp(Kr^\alpha) \text{ for all } r \leq R.$$

148 Theorem E implies the existence of groups of non-uniform exponential growth that
 149 do not imbed uniformly into Hilbert space; this answers a question by Brieussel [15, af-
 150 ter Proposition 2.5], who asked whether there exist groups of non-uniform exponential
 151 growth without the Haagerup property. We also construct groups of non-uniform exponen-
 152 tial growth that admit infinitely many distinct intermediate growth functions at different
 153 scales. Moreover, these examples can be constructed among groups that preform free
 154 groups and groups of intermediate growth.

155 The idea of the proof of Theorem E is as follows. We denote by G_{012} the first Grig-
 156 orchuk group. It acts on the infinite binary tree $\{0, 1\}^*$ and its boundary $\{0, 1\}^\infty$. We
 157 denote by X the orbit $G_{012} \cdot 1^\infty$. We prove in Corollary 7.2 that the group $G \wr_X G_{012}$
 158 has non-uniform exponential growth whenever G is a group of exponential growth. To
 159 prove Corollary 7.2 we show that $G \wr_X G_{012}$ preforms a group of intermediate growth. (In
 160 fact, all known examples of groups of non-uniform exponential growth preform groups of
 161 intermediate growth, though the corresponding group of intermediate growth is not always
 162 given explicitly by their construction ; for more on this see Question 8.7).

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168 2. FIRST PROPERTIES AND EXAMPLES

169 **Lemma 2.1** (A special case of [19, Proposition 2.20]). *The “for some generating set T ”*
 170 *in Definition 1.1 may be changed to “for every generating set T ”.*

171 *Proof.* Assume $G \overset{\sim}{\rightsquigarrow} H$, that T generates H and that $\mathcal{C}(G, S_n)$ coincides with $\mathcal{C}(H, T)$
 172 on ever larger balls. Write $\tau_n : T \rightarrow S_n$ the bijections.

173 Let T' be another generating set of H ; write every $t \in T'$ as a word w_t over T . Let k be
 174 the maximum of the lengths of the w_t . Consider the generating sets $S'_n = \{w_t(\tau_n) : t \in T'\}$
 175 of G obtained by replacing each T -letter in w_t by its corresponding element $\tau_n(t) \in G$.

176 Then, if $\mathcal{C}(G, S_n) \cap B(1, R)$ is isomorphic to $\mathcal{C}(H, T) \cap B(1, R)$, then $\mathcal{C}(G, S'_n) \cap$
 177 $B(1, \lfloor R/k \rfloor)$ is isomorphic to $\mathcal{C}(H, T') \cap B(1, \lfloor R/k \rfloor)$, since they are respective subsets
 178 in the isomorphic graphs $\mathcal{C}(G, S_n) \cap B(1, R)$ and $\mathcal{C}(H, T) \cap B(1, R)$. \square

179 **Lemma 2.2.** *The relation $\overset{\sim}{\rightsquigarrow}$ is a preorder.*

180 *Proof.* It is clear that $G \overset{\sim}{\rightsquigarrow} G$ holds for all groups G .

181 Consider now $G \overset{\sim}{\rightsquigarrow} H \overset{\sim}{\rightsquigarrow} K$, and let U be a generating set for K . There are then
 182 generating sets T_n for H , in bijection with U , such that $\mathcal{C}(H, T_n)$ and $\mathcal{C}(K, U)$ agree in
 183 ever larger balls. For each n , there are generating sets S_{nn} for G , in bijection with T_n ,
 184 such that $\mathcal{C}(G, S_{nn})$ and $\mathcal{C}(H, T_n)$ agree in ever larger balls.

185 Therefore, the generating sets S_{nn} , which are in bijection with U , are such that $\mathcal{C}(G, S_{nn})$
 186 and $\mathcal{C}(K, U)$ agree in ever larger balls, which shows $G \overset{\sim}{\rightsquigarrow} K$. \square

Let \mathbb{F} be the free group on infinitely many generators x_1, x_2, \dots , and consider the space \mathcal{G} of finitely generated groups (G, T) with marked generating set. This marking may be given by a homomorphism $\mathbb{F} \twoheadrightarrow G$ such that almost all x_n map to 1; and this identifies \mathcal{G} with the set of normal subgroups of \mathbb{F} containing almost all the x_n . This turns \mathcal{G} into a locally compact Polish space. In this alternative terminology, we have the obvious

Lemma 2.3. *Let G, H be finitely generated groups. Then $G \overset{\sim}{\rightsquigarrow} H$ if and only if for some (hence all) generating set T , the marked group (H, T) belongs to the closure of $\{(G, S) : S \text{ generates } G\}$ in \mathcal{G} .*

We observe that if $G \overset{\sim}{\rightsquigarrow} H$ and either G or H are finite, then $G = H$. We thus restrict ourselves to infinite, finitely generated groups.

Lemma 2.4. *Let G be a finitely generated group, and let H be a finitely presented group. If $G \overset{\sim}{\rightsquigarrow} H$, then G is a quotient of H .*

Proof. Let T be a generating set of H , and let R be the maximal length of H 's relators in that generating set. If $G \overset{\sim}{\rightsquigarrow} H$, then there exists a generating set S for G such that $\mathcal{C}(G, S)$ and $\mathcal{C}(H, T)$ coincide in a ball of radius R ; so all relations of H hold in T . \square

We note ([19, Example 2.4(e)]) that every residually finite group is a limit of finite groups; however, the closure of the set of finite groups in \mathcal{G} has not been convincingly identified.

It has been shown by Shalom [53] that every group G with Kazhdan's property (T) is a quotient of a finitely presented group with Kazhdan's property (T). Therefore,

Lemma 2.5 ([19, Proposition 2.15]). *If $G \overset{\sim}{\rightsquigarrow} H$ and G does not have Kazhdan's property (T), then neither does H .* \square

There are isolated points in the space of groups; they are studied in [23]. Clearly, isolated groups are minimal elements for $\overset{\sim}{\rightsquigarrow}$; but the converse is not true. For example, \mathbb{Z} and $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ are minimal, but none of them is isolated.

2.1. Partial orders. On some classes of groups, the relation $\overset{\sim}{\rightsquigarrow}$ is also antisymmetric, and therefore defines a partial order. Recall that a group G is Hopfian if every epimorphism $G \twoheadrightarrow G$ is an automorphism.

Lemma 2.6. *Among Hopfian, finitely presented groups, $\overset{\sim}{\rightsquigarrow}$ is an order relation. More generally, if G and H are finitely presented groups with $G \overset{\sim}{\rightsquigarrow} H \overset{\sim}{\rightsquigarrow} G$ and G is Hopfian, then G and H are isomorphic.*

Proof. From $G \overset{\sim}{\rightsquigarrow} H$ and Lemma 2.4 we deduce that G is a quotient of H ; and similarly H is a quotient of G . Therefore we have epimorphisms $G \twoheadrightarrow H \twoheadrightarrow G$, and since G is Hopfian these epimorphisms are isomorphisms. \square

Corollary 2.7. *The relation $\overset{\sim}{\rightsquigarrow}$ is an order relation on polycyclic groups, and on limit groups.*

Proof. Polycyclic groups are known to be finitely presented and residually finite. We will recall some known facts about limit groups in §6.1; for the proof of the corollary it suffices to know that limit groups are residually free and therefore residually finite; and that they are finitely presented.

Since residually finite groups are Hopfian (see [40]), the corollary follows from Lemma 2.6. \square

229 **2.2. Identities and universal statements.** Let G be a group. An *identity* for G is a non-
 230 trivial word $w(x_1, x_2, \dots)$ in the free group on countably many generators, such that
 231 $w(g_1, g_2, \dots) = 1$ for every choice of $g_i \in G$. Note that w is really a word in finitely
 232 many of the x_i 's, namely $w = w(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$.

233 An identity for G is really the following universal sentence. ' $\forall g_1, g_2 \dots (w = 1)$ '. More
 234 generally, any well-formed expression made of conjunctions, disjunctions, equalities, and
 235 universal quantifiers, is a *positive universal sentence*. If furthermore negations are allowed,
 236 it is a *universal sentence*. The *variety* generated by a group G is the set of identities that it
 237 satisfies; and its (*positive*) *universal theory* is the set of (positive) universal sentences that
 238 it satisfies.

239 For example, consider the group $G = \langle x, y, z \mid [x, y]z^{-1}, z^2, [x, z], [y, z] \rangle$. It satisfies
 240 the identity $[x_1, x_2]^2$. It also satisfies the positive universal statement

$$\forall x_1, \dots, x_4 ([x_1, x_2] = 1 \vee [x_1, x_3] = 1 \vee \dots \vee [x_3, x_4] = 1).$$

241 As a last example, limits groups are known to be “commutative-transitive”; this is the
 242 universal statement

$$(1) \quad \forall x, y, z ([x, y] = 1 \wedge [y, z] = 1 \Rightarrow [x, z] = 1).$$

243 Note that this statement is not positive; rewriting it in terms of the primitives \vee, \wedge, \neg
 244 gives $\forall x, y, z (\neg([x, y] = 1 \wedge [y, z] = 1) \vee [x, z] = 1)$. An example of a positive
 245 statement appears in Example 4.13. For more details relating logic to the space of marked
 246 groups, see §6.1 and [19, §5]. In particular, the first assertion of the following lemma
 247 is [19, Proposition 5.2].

248 **Lemma 2.8.** (1) If $G \overset{\sim}{\rightsquigarrow} H$ and G satisfies a universal statement (e.g., an identity),
 249 then H satisfies it too;
 250 (2) If $G \overset{\sim}{\rightsquigarrow} H$ and H is a finitely presented group satisfying a positive universal
 251 statement, then G satisfies it too;
 252 (3) If $G \overset{\sim}{\rightsquigarrow} H$ and G is torsion-free, then H is torsion free. More generally, if F is a
 253 finite subgroup of H , then F imbeds in G .

254 **Remark 2.9.** It is essential not to allow negations in (2): a group with torsion, and more-
 255 over a torsion group, can preform a finitely presented torsion-free group — e.g., Grig-
 256 orchuk's group G preforms \mathbb{F}_3 . In fact, if $G \overset{\sim}{\rightsquigarrow} \mathbb{F}_n$ for some n , then G has the same
 257 positive universal theory as \mathbb{F} . However, G is universally equivalent to H if and only if G
 258 is a non-abelian limit group of Sela (see §6.1, that is, if $\mathbb{F} \overset{\sim}{\rightsquigarrow} G$).

259 The lemma implies in particular that if G is virtually nilpotent, then every group in the
 260 same connected component has the same language of positive universal statements. How-
 261 ever, in any such connected component there are groups that are not universally equivalent
 262 to G .

263 *Proof.* Ad (1): consider a universal statement satisfied in G ; it is of the form $\forall x_1, \dots, x_n (E)$
 264 for a boolean expression E made of identities w_1, \dots, w_ℓ . Let R be the maximal length
 265 w_1, \dots, w_ℓ .

266 Consider arbitrary $h_1, \dots, h_n \in H$. Extend $\{h_1, \dots, h_n\}$ to a generating set T of H ,
 267 and find a generating set S of G such that the balls of radius R in $\mathcal{C}(G, S)$ and $\mathcal{C}(H, T)$
 268 coincide. Let g_1, \dots, g_n be the generators of G that correspond to h_1, \dots, h_n respectively.
 269 Then w_i traces a path in $\mathcal{C}(G, S)$ that remains in an R -neighbourhood of the origin, so
 270 w_i traces a closed loop in $\mathcal{C}(G, S)$ if and only if it traces a closed loop in $\mathcal{C}(H, T)$;
 271 therefore, $w_i(h_1, \dots, h_n) = 1 \Leftrightarrow w_i(g_1, \dots, g_n) = 1$, so $E(h_1, \dots, h_n)$ follows from
 272 $E(g_1, \dots, g_n)$.

Ad (2): Lemma 2.4 shows that G is a quotient of H ; and positive universal statements are preserved by taking quotients.

Ad (3): consider a finite group F . Then the fact that F is *not* a subgroup of G is a universal statement: writing f_1, \dots, f_k the elements of F , with multiplication table $f_i f_j = f_{m(i,j)}$, the statement is $\forall g_1, \dots, g_k (g_i = g_j \text{ for some } i \neq j \vee g_i g_j \neq g_{m(i,j)} \text{ for some } i, j)$. Therefore (3) follows from (1). \square

2.3. Varieties. We defined varieties in §2.2 as collections of identities. Alternatively (see [44]), it is a family of groups closed under taking subgroups, quotients and cartesian products, namely the class \mathcal{V} of all the groups that satisfy these identities. The variety \mathcal{V} is *finitely based* if it may be defined by finitely many identities. It is *finite* if all finitely generated groups in the variety are finite. For a group G , one defines $\mathcal{V}(G) = \langle w_i(g_1, g_2, \dots) : i \geq 1, g_1, g_2, \dots \in G \rangle$, the *verbal subgroup* of G corresponding to \mathcal{V} ; thus $\mathcal{V}(G) = 1$ if and only if G belongs to the variety. The k -generated *relatively free group* is $\mathbb{V}_k := \mathbb{F}_k / \mathcal{V}(\mathbb{F}_k)$; it belongs to \mathcal{V} , and every k -generated group in \mathcal{V} is a quotient of \mathbb{V}_k . A direct consequence of Lemma 2.8(1) is the

Lemma 2.10. *If $G \overset{\sim}{\sim} H$ and G belongs to \mathcal{V} , then H belongs to \mathcal{V} .* \square

We will consider, in later sections, the restriction of the relation $\overset{\sim}{\sim}$ to groups belonging to a variety. Just as \mathcal{G} is a topology on the normal subgroups of \mathbb{F}_k , there is a topology $\mathcal{G}(\mathcal{V})$ on the normal subgroups of \mathbb{V}_k , or equivalently on the normal subgroups of \mathbb{F}_k that contain $\mathcal{V}(\mathbb{F}_k)$. Directly from the definitions,

Lemma 2.11 ([19, Lemma 2.2]). *The natural map $\mathcal{G}(\mathcal{V}) \hookrightarrow \mathcal{G}$ is a homeomorphism on its image, and that the image is closed if and only if \mathbb{V}_k is finitely presented for all $k \in \mathbb{N}$.* \square

Lemma 2.12. *Let \mathcal{V} be a finite variety. If $G \overset{\sim}{\sim} H$, then $\mathcal{V}(G) \overset{\sim}{\sim} \mathcal{V}(H)$.*

Proof. Let H be generated by a set $T = \{h_1, \dots, h_k\}$ of cardinality k , and let \mathbb{F}_k denote the free group on k generators x_1, \dots, x_k . Then $\mathcal{V}(\mathbb{F}_k)$ admits a generating set of the form $w(v_1, \dots)$ for some identities w in \mathcal{V} and some $v_1, \dots \in \mathbb{F}_k$. Then $\mathcal{V}(H)$ is generated by the set T' of all corresponding $w(v_1(h_1, \dots, h_k), \dots)$.

Consider a generating set $S = \{g_1, \dots, g_k\}$ of G , such that $\mathcal{C}(G, S)$ coincides with $\mathcal{C}(H, T)$ in a large ball; then $S' = \{w(v_1(g_1, \dots, g_k), \dots), \dots\}$ generates $\mathcal{V}(G)$, and the Cayley graphs $\mathcal{C}(\mathcal{V}(G), S')$ coincides with $\mathcal{C}(\mathcal{V}(H), T')$ in a large ball. \square

Given a variety \mathcal{V} , the *verbal product* of groups G_1, G_2, \dots, G_n is defined as follows: first set $G = G_1 * G_2 \cdots * G_n$ the free product; then

$$\prod_{\mathcal{V}} G_i = \frac{G}{\mathcal{V}(G) \cap \langle [g_i, g_j] : g_i \in G_i^G, g_j \in G_j^G, i \neq j \rangle}.$$

For example, if \mathcal{V} is the variety of all groups, then $\prod_{\mathcal{V}}$ is the free product; while if \mathcal{V} is the variety of abelian groups, then $\prod_{\mathcal{V}}$ is the direct product.

Recall that the *wreath product* of two groups G_1, G_2 is

$$G_1 \wr G_2 = \{f : G_2 \rightarrow G_1 \text{ of finite support}\} \rtimes G_2,$$

where G_2 acts by shift on functions $G_2 \rightarrow G_1$.

Lemma 2.13. *Let G_1, G_2, H_1, H_2 be groups, and assume $G_1 \overset{\sim}{\sim} H_1$ and $G_2 \overset{\sim}{\sim} H_2$. Then*

- (1) $G_1 \times G_2 \overset{\sim}{\sim} H_1 \times H_2$;
- (2) $G_1 * G_2 \overset{\sim}{\sim} H_1 * H_2$;
- (3) *Let \mathcal{V} be a variety of groups. Then $\prod_{\mathcal{V}} G_i \overset{\sim}{\sim} \prod_{\mathcal{V}} H_i$;*

313 (4) $G_1 \wr G_2 \overset{\sim}{\rightsquigarrow} H_1 \wr H_2$.

314 *Proof.* We start by (2), and argue that, for arbitrarily large R , we can make balls of radius
 315 R agree in respective Cayley graphs. For all $i \in \{1, 2\}$, let T_i generate H_i , and let S_i
 316 generate G_i in such a manner that balls of radius R coincide in $\mathcal{C}(G_i, S_i)$ and $\mathcal{C}(H_i, T_i)$.
 317 Then $T := \sqcup T_i$ generates $H := *_i H_i$, and the corresponding set $S := \sqcup S_i$ generates
 318 $*_i G_i$. Balls of radius R coincide in $\mathcal{C}(G, S)$ and $\mathcal{C}(H, T)$.

319 Ad (3), the relations imposed on $*_i G_i$ and $*_i H_i$ are formally defined by \mathcal{V} , so again
 320 balls of radius R in $\mathcal{C}(\prod_{\mathcal{V}} G_i, S)$ and $\mathcal{C}(\prod_{\mathcal{V}} H_i, T)$ coincide.

321 (1) is a special case of (3).

322 Ad (4), note that the relations giving $G_1 \wr G_2$ from $G := G_1 * G_2$ are $[x_1^{x_2}, y_1^{y_2}]$ for all
 323 $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2 \setminus \{1\}$. These relations do not exactly define a varietal product;
 324 but nevertheless there is a bijection between non-trivial elements of norm $\leq R$ in G_2 and
 325 H_2 , and between elements of norm $\leq R$ in G_1 and H_1 . The result again follows. \square

326 Note that in (1) we can have $G_1 \times C \overset{\sim}{\rightsquigarrow} H_1 \times C$ without having $G_1 \overset{\sim}{\rightsquigarrow} H_1$. We we
 327 examine more carefully this for abelian groups in §3:

328 **Example 2.14.** We have $1 \times \mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z} \times \mathbb{Z}$, yet 1 doesn't preform \mathbb{Z} .

329 For $A = \mathbb{Z}/6 \times \mathbb{Z}$, $B = \mathbb{Z}/35 \times \mathbb{Z}$, $C = \mathbb{Z}/10 \times \mathbb{Z}$, $D = \mathbb{Z}/21 \times \mathbb{Z}$, we also have
 330 $A \times B \overset{\sim}{\rightsquigarrow} C \times D$ while A, B, C, D are mutually incomparable.

331 *Proof.* Consider $\{(1, 0), (0, 1)\}$ a generating set of $\mathbb{Z} \times \mathbb{Z}$, and, for arbitrary $R \in \mathbb{N}$, the
 332 generating set $\{(0, 1), (0, 2R + 1)\}$ of $1 \times \mathbb{Z}$. Their Cayley graphs agree on a ball of radius
 333 R .

334 For the second claim, note that $A \times B$ is isomorphic to $C \times D$, but for any two groups
 335 among A, B, C, D , none is a quotient of the other. \square

336 Similarly, in (2) we can have $G_1 * C \overset{\sim}{\rightsquigarrow} H_1 * C$ without having $G_1 \overset{\sim}{\rightsquigarrow} H_1$. We will
 337 examine more closely the situation of free groups in §6.1; here and in the sequel we use
 338 the notation \mathbb{F}_k for free groups on k generators. For now, we just mention the

339 **Example 2.15.** Let G be a k -generated group. Then, for every $m \geq 2$, the free product
 340 $G * \mathbb{F}_m$ preforms $\mathbb{F}_{k+m} = \mathbb{F}_k * \mathbb{F}_m$; yet G need not preform \mathbb{F}_k , for example if G satisfies
 341 an identity.

342 *Proof.* Let S generate \mathbb{F}_k , let T generate \mathbb{F}_m , and let $\{g_1, \dots, g_k\}$ generate G . Then $S \sqcup T$
 343 generates $\mathbb{F}_k * \mathbb{F}_m$. In \mathbb{F}_m , there exist elements w_1, \dots, w_k such that no relation among
 344 them and T , of length $\leq R$, holds; consider the generating set $\{g_1 w_1, \dots, g_k w_k\} \sqcup T$ of
 345 $G * \mathbb{F}_m$. Then no relation of length $\leq R$ holds among them. \square

346 Note finally that in (4) we may have $G_1 \wr C \overset{\sim}{\rightsquigarrow} H_1 \wr C$ without having $G_1 \overset{\sim}{\rightsquigarrow} H_1$;
 347 see §6.5 for more examples:

348 **Example 2.16.** Consider A, B arbitrary groups, and an infinite group C . Then $(A * B) \wr$
 349 $C \overset{\sim}{\rightsquigarrow} (A \times B) \wr C$.

350 On the other hand, if A and B are non-trivial, finitely presented, and each satisfies an
 351 identity, then $A * B$ does not satisfy the identities of $A \times B$, so $A * B$ doesn't preform
 352 $A \times B$ by Lemma 2.8(2).

353 *Proof.* Let S, T, U be generating set of A, B, C respectively. Then, as generating set of
 354 $(A \times B) \wr C$, we consider $S' \sqcup T' \sqcup U$, in which S' corresponds to the generators of A
 355 supported at $1 \in C$, and similarly for T' .

For arbitrary $R \in \mathbb{N}$, choose $x \in C$ of norm $> R$, and consider the following generating set $S'' \sqcup T'' \sqcup U$ of $(A * B) \wr H$. The copy S'' of S corresponds to the generators of A supported at $1 \in C$, while the copy T'' corresponds to the generators of T supported at x .

Both $(A \times B) \wr C$ and $(A * B) \wr C$ are quotients of $A * B * C$; in both cases, all relations of the form $[s_1^h, s_2]$ and $[t_1^h, t_2]$ are imposed for all $h \neq 1$ and $s_i \in S', t_i \in T'$, respectively $s_i \in S'', t_i \in T''$. However, in the former case, all relations of the form $[s^h, t]$ are also imposed for all $h \in H$ and $s \in S', t \in T'$. In the latter case, these relations are only imposed for $h \neq x$ and $s \in S'', t \in T''$. However, this distinction is invisible in the ball of radius R . \square

2.4. Limits and prelimits of groups with a given subgroup or quotient. We start by the following straightforward lemma.

Lemma 2.17. *If $A \overset{\sim}{\hookrightarrow} B$ and A is a subgroup of G , then there exists a group H containing B as a subgroup and satisfying $G \overset{\sim}{\hookrightarrow} H$:*

$$\begin{array}{ccc} G & \overset{\sim}{\hookrightarrow} & H \\ \cup & & \cup \\ A & \overset{\sim}{\hookrightarrow} & B. \end{array}$$

Proof. Consider finite generating sets S_n of A and T of B such that (A, S_n) converges to (B, T) in the space \mathcal{G} of marked groups, as $n \rightarrow \infty$. Let S be a finite generating set of G . Set $S'_n = S \sqcup S_n$; these define finite generating sets of G . Consider a subsequence (n_k) such that (G, S'_{n_k}) converges in \mathcal{G} ; denote its limit by $(H, U \sqcup V)$.

In particular, (A, S_{n_k}) converges to the subgroup $\langle V \rangle$ of H . Since (A, S_n) converges to B , we conclude that $\langle V \rangle$ is isomorphic to B . \square

Lemma 2.18. *If $A \overset{\sim}{\hookrightarrow} B$ and A is a quotient of G , then there exists a group H with $G \overset{\sim}{\hookrightarrow} H$ and B is a quotient of H :*

$$\begin{array}{ccc} G & \overset{\sim}{\hookrightarrow} & H \\ \downarrow & & \downarrow \\ A & \overset{\sim}{\hookrightarrow} & B. \end{array}$$

Proof. Let A, B be k -generated, with T a generating set for B . Since A preforms B , there exists a sequence of generating sets S_n of cardinality k such that $(A, S_n) \rightarrow (B, T)$. Without loss of generality, we may assume $1 \in S_n$ for all $n \in \mathbb{N}$.

Let $\pi: G \twoheadrightarrow A$ be the given epimorphism. Let G be ℓ -generated. Then for each $n \in \mathbb{N}$ there exists a generating set $S'_n = S''_n \sqcup S'''_n$ of G such that S''_n maps bijectively to S_n under π and S'''_n maps to $1 \in A$ and has cardinality ℓ . Indeed first choose a generating set S' for G of cardinality ℓ ; then, for each $n \in \mathbb{N}$, choose an arbitrary lift S'_n of S_n ; and multiply each $g \in S'$ by an appropriate word in S'_n to obtain S'''_n mapping to 1 .

Passing if need be to a subsequence, we can assume that (G, S'_n) converges in the space \mathcal{G} of marked groups. Denote the limit of the subsequence by (H, T') , again with decomposition $T' = T'' \sqcup T'''$. Let us construct an epimorphism $\rho: H \twoheadrightarrow B$, showing that B is a quotient of H . Recall that T'' is naturally in bijection with T , via S''_n and S_n . We define ρ on T'' by this bijection, and put $\rho(t) = 1$ for all $t \in T'''$.

To prove that ρ is a homomorphism, consider a word $w(x_1, \dots, x_{k+\ell})$ with $w(T') = 1$ in H . Since (G, S'_n) converges to (H, T') , for sufficiently large $n \in \mathbb{N}$ we have $w(S'_n) = 1$ in G . Let $v(x_1, \dots, x_k)$ denote the word obtained from w by deleting its letters $x_{k+1}, \dots, x_{k+\ell}$. Since π is a homomorphism, we then have $v(S_n) = 1$, and therefore in the limit $v(T) = 1$. This is precisely the result of computing $\rho(w(T'))$ letter by letter.

395 Finally, T is in the image of ρ so ρ is surjective. \square

396 We may improve on Lemma 2.18 in case the quotient is by a verbal subgroup:

397 **Lemma 2.19.** *Let the group G be generated by a set of cardinality k , and let \mathcal{V} be a variety.*
 398 *If $G/\mathcal{V}(G) \overset{\sim}{\hookrightarrow} \mathbb{V}_k$, then there exists a group H with $G \overset{\sim}{\hookrightarrow} H$ and $\mathbb{V}_k = H/\mathcal{V}(H)$:*

$$\begin{array}{ccc} G & \overset{\sim}{\hookrightarrow} & H \\ \downarrow & & \downarrow \\ G/\mathcal{V}(G) & \overset{\sim}{\hookrightarrow} & \mathbb{V}_k. \end{array}$$

399 *Proof.* We proceed first as in the proof of Lemma 2.18, to construct a group H and an
 400 epimorphism $\rho : H \twoheadrightarrow \mathbb{V}_k$.

401 On the one hand, $\mathcal{V}(H) \subseteq \ker \rho$, because \mathbb{V}_k belongs to \mathcal{V} . On the other hand, consider
 402 $c \in \ker \rho$, and write $c = w(T)$ as a word in the generators T of H . Then $\rho(w(T)) = 1$, so
 403 w belongs to the variety $\mathcal{V}(\mathbb{F}_k)$ because \mathbb{V}_k is relatively free. It follows that c belongs to
 404 $\mathcal{V}(H)$. \square

405 **Lemma 2.20** ([19, Proposition 2.25]). *If $G \overset{\sim}{\hookrightarrow} H$ and A is a quotient of G , then there*
 406 *exists a group B with $A \overset{\sim}{\hookrightarrow} B$ and B is a quotient of H :*

$$\begin{array}{ccc} G & \overset{\sim}{\hookrightarrow} & H \\ \downarrow & & \downarrow \\ A & \overset{\sim}{\hookrightarrow} & B. \end{array}$$

407 Let us turn to the converse property: if $A \overset{\sim}{\hookrightarrow} B$ and B is a subgroup of H , does there
 408 exist a group G containing A with $G \overset{\sim}{\hookrightarrow} H$? Given a subgroup B of a group H , we say
 409 that the pair (H, B) satisfies the “prelimit of an overgroup” property if, whenever A is a
 410 group which preforms B , there exists a group G which preforms H and contains A :

$$\begin{array}{ccc} \exists G & \overset{\sim}{\hookrightarrow} & H \\ \cup & & \cup \\ \forall A & \overset{\sim}{\hookrightarrow} & B. \end{array}$$

411 We then say that H has the “prelimit of an overgroup” property if (H, B) has that property
 412 for all finitely generated subgroups B of H .

413 **Question 2.21.** Which finitely generated groups have the “prelimit of an overgroup” prop-
 414 erty?

415 It is clear that if H has very few subgroups, for example if every proper subgroup of H
 416 is finite, then H has the “prelimit of an overgroup” property.

417 **Lemma 2.22.** *All finitely generated abelian groups have the “prelimit of an overgroup”*
 418 *property.*

419 *Proof.* Inclusions of finitely generated abelian groups into one another can be decomposed
 420 into the following “elementary inclusions”: $B \subseteq B \oplus \mathbb{Z}$, $B \subseteq B \oplus \mathbb{Z}/a\mathbb{Z}$ and $B \oplus$
 421 $\mathbb{Z}/a\mathbb{Z} \subseteq B \oplus \mathbb{Z}/ab\mathbb{Z}$. Similarly, the cases to consider for A that preforms B are of the
 422 form $\mathbb{Z} \oplus \mathbb{Z}/ac\mathbb{Z} \overset{\sim}{\hookrightarrow} \mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z} \overset{\sim}{\hookrightarrow} \mathbb{Z}^2$. To prove the lemma, it suffices therefore
 423 to consider the following case: $B = \mathbb{Z}^2 \oplus \mathbb{Z}/a\mathbb{Z}$ is a subgroup of $H = \mathbb{Z}^2 \oplus \mathbb{Z}/ab\mathbb{Z}$, and
 424 $A = \mathbb{Z} \oplus \mathbb{Z}/ac\mathbb{Z}$ preforms B . We observe that in this case $G := \mathbb{Z} \oplus \mathbb{Z}/abc\mathbb{Z}$ contains A ,
 425 and preforms H . \square

Example 2.23 (Groups without the “prelimit of an overgroup” property). *There are finitely generated groups $A \preccurlyeq B \subseteq H$ such that there exists no group G with $A \subseteq G \preccurlyeq H$.*

Take indeed $A = \mathbb{F}_2 \wr \mathbb{Z}$; it preforms $B = \mathbb{Z}^2 \wr \mathbb{Z}$, which is metabelian. By [12], every metabelian group imbeds in a finitely presented metabelian group H . If $G \preccurlyeq H$, then G is a quotient of H . This shows that every group which preforms H is metabelian. Therefore, there are no groups that preform H that contain A as a subgroup.

Example 2.24 (Finitely presented groups without the “prelimit of an overgroup” property). *Here is another example of this kind. Consider a finitely presented infinite torsion-free simple group H containing a non-abelian free group $B = \mathbb{F}_3$ as a subgroup; such groups do exist, see [17]. Set $A = \mathbb{F}_2 \times \mathbb{Z}/2\mathbb{Z}$; then $A \preccurlyeq B$ and $B \subseteq H$. However, if $G \preccurlyeq H$, then $G = H$ because H is finitely presented and simple. However, H does not contain A because H is torsion-free.*

It is usually not true that, if G preforms H , then the torsion of G and H coincide. Here is a partial result in this direction:

Lemma 2.25. *Let G and H be groups with H finitely presented and $G \preccurlyeq H$, and let \mathcal{V} be a variety. Then*

- (1) $\#\mathcal{V}(G) = \#\mathcal{V}(H)$;
- (2) if $\mathcal{V}(G)$ is finite, then $\mathcal{V}(G)$ is isomorphic to $\mathcal{V}(H)$.

Proof. By Lemma 2.8(1), the group G is a quotient of H , so $\mathcal{V}(G)$ is a quotient of $\mathcal{V}(H)$. In particular, $\#\mathcal{V}(G) \leq \#\mathcal{V}(H)$. Furthermore, if $\mathcal{V}(H)$ is finite then Lemma 2.8(3) implies that $\mathcal{V}(G)$ and $\mathcal{V}(H)$ are isomorphic. It therefore remains to prove $\#\mathcal{V}(G) \geq \#\mathcal{V}(H)$. We will prove in fact that, if $\#\mathcal{V}(H) \geq N$, then $\#\mathcal{V}(G) \geq N$.

Choose generating sets S_n of G and T of H , of cardinality k , such that (G, S_n) converges to (H, T) is the space \mathcal{G} of marked groups.

Consider then N distinct elements h_1, \dots, h_N in $\mathcal{V}(H)$, and write each $h_j = w_j(T)$ for a word $w_j \in \mathcal{V}(\mathbb{F}_k)$. Take $R \in \mathbb{N}$ bigger than the length of each w_j , and let i be such that the balls of radius R in $\mathcal{C}(G, S_i)$ and $\mathcal{C}(H, T)$ coincide. Then the ball of radius R in $\mathcal{C}(H, T)$ contains at least the N distinct elements h_1, \dots, h_N from $\mathcal{V}(H)$, so the ball of radius R in $\mathcal{C}(G, S_n)$ also contains at least N distinct elements $w_1(S_n), \dots, w_N(S_n)$ from $\mathcal{V}(G)$. \square

2.5. Universal theories of solvable groups. For a group G , we denote by $G^{(n)}$ its derived series, with $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. In particular $G^{(1)} = G'$ and $G^{(2)} = G''$.

Here is an example of metabelian group that preforms the free group in its variety. In the next sections, we will study when a nilpotent group preforms the free group in the variety it generates.

Example 2.26. *We have $\mathbb{Z} \wr \mathbb{Z} \preccurlyeq \mathbb{F}_2/\mathbb{F}_2''$.*

Proof. Consider the presentation $\langle a, t \mid [a, a^{t^m}] \forall m \rangle$ of $\mathbb{Z} \wr \mathbb{Z}$, and its generating sets $S_n = \{t, t^n a\}$. Write $u = at^n$; then $[t, u] = [a, t]$, and $[t, u]^{t^x u^y}$ all have distinct supports, for $|x|, |y| \leq n$. \square

Chapuis considers in [20] the universal theory of some solvable groups; he shows that $\mathbb{F}_k/\mathbb{F}_k''$ and $\mathbb{Z}^k \wr \mathbb{Z}^\ell$ have the same universal theory. An explicit description of that theory is given in [21]. On the other hand, $\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}$ and $\mathbb{F}_k/\mathbb{F}_k^{(3)}$ do not have the same theory.

Timoshenko proves in [55] that, if G_1, G_2 have the same universal theory, and H_1, H_2 have the same universal theory, then $G_1 \wr H_1$ and $G_2 \wr H_2$ have the same universal theory. He shows, however, that the varietal wreath product does not, in general, enjoy this property; in particular, it fails in the metabelian variety [57].

He also shows in [56] that, if G is the quotient of $\mathbb{S}_{2,n} := \mathbb{F}_2/\mathbb{F}_2^{(n)}$ by a finitely generated normal subgroup, and has the same universal theory as $\mathbb{S}_{2,n}$, then either $G \cong \mathbb{S}_{2,n}$ or G is a verbal wreath product $\mathbb{Z} \wr \mathbb{Z}$, in the variety of soluble groups of class $n - 1$. He shows:

Lemma 2.27. *Let \mathbb{V} be a free group in a variety \mathcal{V} , and let H be a subgroup of \mathbb{V} that generates the same variety \mathcal{V} . Assume that \mathbb{V} is discriminating (see §4.1). Then the universal theories of \mathbb{V} and of H coincide.*

Lemma 2.28. *Let G, H be groups in a variety \mathcal{V} , and assume that G is universally equivalent to H . Then A is residually B .*

Timoshenko also considered the universal theories of partially commutative metabelian groups in [58] and subsequent papers.

3. ABELIAN GROUPS

By Corollary 2.7, the relation \lesssim is a partial order on the set of abelian groups. The following is straightforward.

Lemma 3.1. *For non-zero $m, n \in \mathbb{N}$, we have $\mathbb{Z}^m \lesssim \mathbb{Z}^n$ if and only if $m \leq n$.*

Proof. If $\mathbb{Z}^m \lesssim \mathbb{Z}^n$, then \mathbb{Z}^m is a quotient of \mathbb{Z}^n by Lemma 2.4, so $m \leq n$. Conversely, if $m \leq n$, then choose for \mathbb{Z}^n a basis T as generating set, and let $\{e_1, \dots, e_m\}$ be a basis of \mathbb{Z}^m . For arbitrary $R \in \mathbb{N}$, choose $S = \{e_1, \dots, e_m, Re_1, R^2e_1, \dots, R^{n-m}e_1\}$ as generating set for \mathbb{Z}^m , and note that $\mathcal{C}(\mathbb{Z}^m, S)$ and $\mathcal{C}(\mathbb{Z}^n, T)$ agree on a ball of radius R . \square

We now show that all infinite abelian groups are in the same component of \lesssim , which has diameter 2; more precisely,

Proposition 3.2. *The restriction of \lesssim to infinite abelian subgroups is a net: a partial order in which every pair of elements has an upper bound.*

Proposition 3.3. *For infinite abelian finitely generated groups A, B , we have $A \lesssim B$ if and only if A is a quotient of B via a map $B \twoheadrightarrow A$ that is injective on the torsion of B .*

Proof. If $A \lesssim B$, then A is a quotient of B by Lemma 2.4. Let R be larger than the order of the torsion of A and B , and let S, T be generating sets of A, B respectively such that $\mathcal{C}(A, S)$ and $\mathcal{C}(B, T)$ coincide in the ball of radius R . Then all torsion elements of B belong to that ball, and are mapped, by the identification of the ball, to torsion elements of A . This imbeds the torsion of B into that of A .

Conversely, consider an epimorphism $B \twoheadrightarrow A$ that is injective on the torsion of B . Let $B = G_0 \twoheadrightarrow G_1 \twoheadrightarrow \dots \twoheadrightarrow G_n = A$ be a maximal sequence of non-invertible epimorphisms. If we prove $G_i \lesssim G_{i-1}$ for all $i = 1, \dots, n$, then we have $A \lesssim B$ by Lemma 2.2, so we may restrict to a minimal epimorphism $\pi: B \twoheadrightarrow A$. Its kernel is thus infinite cyclic, and we have reduced to the case $A = \mathbb{Z} \oplus \mathbb{Z}/(k\ell)\mathbb{Z}$ and $B = \mathbb{Z}^2 \oplus \mathbb{Z}/k\mathbb{Z}$.

In that case, we consider $T = \{f_1, f_2, f_3\}$ the standard generating set for B , and denote by $\{e_1, e_2\}$ the standard generators for A . For arbitrary $R \in \mathbb{N}$, we consider the generating set $S = \{\ell e_1, e_2, e_1 + Re_2\}$ for A , and note that the balls of radius R in $\mathcal{C}(B, T)$ and $\mathcal{C}(A, S)$ coincide. \square

512 *Proof of Proposition 3.2.* Consider A, B abelian groups, written as

$$A = \bigoplus_{i=1}^a \mathbb{Z}/m_i \mathbb{Z}, \quad B = \bigoplus_{i=1}^b \mathbb{Z}/n_i \mathbb{Z}.$$

513 Then both groups preform $\mathbb{Z}^{\max(a,b)}$. \square

514 **Corollary 3.4.** *Let A be an infinite abelian group. Then A is torsion-free if and only if the*
 515 *set of groups that are preformed by A is linearly ordered.*

516 *Proof.* If $A = \mathbb{Z}^d$ and $A \preccurlyeq B$, then $B = \mathbb{Z}^e$ for some $e \geq d$. The set of such B is
 517 order-isomorphic to $\{d, d+1, \dots\}$.

518 Now suppose that A is not torsion-free. By Proposition 3.3, we have $A \preccurlyeq \mathbb{Z}^d \oplus \mathbb{Z}/p\mathbb{Z}$
 519 for some $p > 1$ and $d > 1$. Then $A \preccurlyeq \mathbb{Z}^{d+1}$ and $A \preccurlyeq \mathbb{Z}^{d+1} \oplus \mathbb{Z}/p\mathbb{Z}$, but these last groups
 520 are not comparable. \square

521 Let us denote by \mathcal{A} the subset of \mathcal{G} consisting of abelian groups, and by \mathcal{A}/\cong the set
 522 of isomorphism classes of abelian groups; as we noted above, $(\mathcal{A}/\cong, \preccurlyeq)$ is a net.

523 **Corollary 3.5.** *Every finite partial order is imbeddable in $(\mathcal{A}/\cong, \preccurlyeq)$.*

524 *Proof.* Let (X, \leq) be a partially ordered set. We identify $x \in X$ with $I_x := \{z \in X : z \geq$
 525 $x\}$, and have $I_y \subseteq I_x \Leftrightarrow x \leq y$; therefore, we assume without loss of generality that X is
 526 contained, for some $N \in \mathbb{N}$, in the partially ordered set of subsets of $\{1, \dots, N\}$, ordered
 527 under reverse inclusion.

528 Consider N distinct prime numbers p_1, \dots, p_N . For any subset $U \subseteq \{1, \dots, N\}$, con-
 529 sider the $N+1$ -generated group A_U defined by

$$A_U = \bigoplus_{i \in U} \mathbb{Z}/p_i \mathbb{Z} \oplus \mathbb{Z}^{1+N-\#U}.$$

530 Observe that the torsion subgroup of A_U is contained in the torsion group of $A_{U'}$ if and
 531 only if $U' \subseteq U$. Observe also that if $U' \subseteq U$, then A_U is a quotient of $A_{U'}$. By Proposi-
 532 tion 3.3, we get $A_U \preccurlyeq A_{U'}$ if and only if $U' \subseteq U$. \square

533 **Remark 3.6.** Some countable orders cannot be imbedded in $(\mathcal{A}/\cong, \preccurlyeq)$; for example,
 534 $\mathbb{N} \cup \{\infty\}$. Observe indeed that a countable increasing sequence of non-isomorphic abelian
 535 groups has no common upper bound in $(\mathcal{A}/\cong, \preccurlyeq)$.

536 **Proposition 3.7.** *The group of order-preserving bijections of $(\mathcal{A}/\cong, \preccurlyeq)$ is the infinite*
 537 *symmetric group on a countable set. If we identify this countable set with the prime num-*
 538 *bers, then the action on infinite abelian groups is as follows. A permutation $p \mapsto \sigma(p)$ of*
 539 *the primes acts as*

$$(2) \quad \mathbb{Z}^d \oplus \mathbb{Z}/p_1^{\nu_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k^{\nu_k} \mathbb{Z} \mapsto \mathbb{Z}^d \oplus \mathbb{Z}/\sigma(p_1)^{\nu_1} \mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\sigma(p_k)^{\nu_k} \mathbb{Z}.$$

540 *Proof.* As a countable set, we take the set \mathcal{P} of prime numbers. By Proposition 3.3, the
 541 group \mathfrak{S} of permutations of \mathcal{P} acts on $(\mathcal{A}/\cong, \preccurlyeq)$ by (2). We wish to prove that there are
 542 no other order-preserving bijections. We implement this in the following lemmas.

543 **Lemma 3.8.** *Every order-preserving bijection of infinite abelian groups fixes torsion-free*
 544 *abelian groups.*

545 *Proof.* By Corollary 3.4, torsion-free abelian groups are characterized by the fact that the
 546 set of groups that they preform is linearly ordered. Let ϕ be an order-preserving bijection.
 547 Observe that ϕ must fix the minimal element \mathbb{Z} . Note that groups that are preformed by \mathbb{Z}

are linearly ordered by \mathbb{N} , so admit no order isomorphism. Therefore, $\phi(\mathbb{Z}^d) = \mathbb{Z}^d$ for any $d \geq 1$. \square

Lemma 3.9. *Every order-preserving bijection of infinite abelian groups preserves the number of factors in a minimal decomposition as a product of cyclic groups.*

Proof. Consider an infinite abelian group A , and let ℓ be the minimal number of cyclic subgroups in the decomposition of A in a product of (finite or infinite) cyclic groups. Since A is infinite, at least one subgroup in the decomposition is infinite. We know that for any $p \in \mathbb{N}$ the group $\mathbb{Z} + p\mathbb{Z}$ preforms \mathbb{Z}^2 , so A preforms \mathbb{Z}^ℓ .

Observe also that for $k < \ell$ the group A cannot be generated by k elements, so A is not a quotient of \mathbb{Z}^k . By Proposition 3.3, A doesn't preform \mathbb{Z}^k for $k < \ell$.

Let ϕ be an order-preserving bijection. By Lemma 3.8, we have $\phi(\mathbb{Z}^k) = \mathbb{Z}^k$ for all $k \geq 1$, so $\phi(A)$ preforms \mathbb{Z}^ℓ but not \mathbb{Z}^k for $k < \ell$. Therefore, $\phi(A)$ requires precisely ℓ factors in a minimal decomposition as a product of cyclic groups. \square

Lemma 3.10. *Every order-preserving bijection ϕ of infinite abelian groups preserves the number of finite and infinite factors in a minimal decomposition as a product of cyclic groups.*

Proof. Let A be an infinite abelian group. Let t be the minimal number of finite cyclic groups in its decomposition into a product of cyclic ones, and let $t + d$ be the minimal total number of finite cyclic groups in such decomposition. We have $A = \mathbb{Z}^d \oplus \bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$, with $n_i \geq 2$. Observe that A is preformed by $\mathbb{Z} \oplus \bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$, and thus is preformed by some group whose minimal total number of cyclic groups in a decomposition equals $t + 1$. Observe then that A is not preformed by any group for which this minimal number is $\leq t$. Indeed, if B preforms A , then B is an infinite group, so the number of infinite cyclic group in the decomposition is ≥ 1 . We know that the torsion subgroup of A , that is $\bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$, is a subgroup of the torsion subgroup of B . Therefore, the minimal number of finite cyclic groups in the decomposition of B is at least t . The statement of the lemma now follows from the previous lemma. \square

Consider now an order-preserving bijection ϕ of abelian groups. Let us show that for every prime p there exists a prime q such that $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$. First observe that any group with non-trivial torsion and which preforms \mathbb{Z}^2 has the form $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$. If n is not a prime number, then n can be written as $n = n_1 n_2$ with $n_1, n_2 \geq 2$ and in this case $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ preforms $\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$. This implies that the groups of the form $A = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ are characterized by the following properties: A is not torsion-free; A preforms \mathbb{Z}^2 ; if B is such that $A \rightsquigarrow B \rightsquigarrow \mathbb{Z}^2$ then either $B = A$ or $B = \mathbb{Z}^2$. This implies that $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ for some prime q .

As we have already mentioned, every permutation of the primes induces an order-preserving permutation of infinite abelian group. It remains to prove that a permutation of infinite abelian groups is determined by its action on groups of the form $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Consider therefore such a permutation ϕ , and assume that it fixes $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ for all $p \in \mathcal{P}$. We wish to show that it fixes every abelian group.

Lemma 3.11. *Let ϕ be an order-preserving bijection of the infinite abelian groups, such that $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ for all primes p .*

Then for all $k, m \geq 1$ we have $\phi(\mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}) = \mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}$.

Proof. Set $A = \mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}$. By Lemma 3.10, we have $\phi(A) = \mathbb{Z}^k \oplus \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$. We proceed by induction on m to show that A is fixed.

593 If $m = 1$, then A is preformed by $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ which is fixed, so $\phi(A)$ is also preformed
 594 by this group, and $n|p$. Since $n \neq 1$, we have $n = p$ as required.

595 Consider then $m \geq 2$. We have $A \overset{\sim}{\sim} \mathbb{Z}^{k+1} \oplus \mathbb{Z}/p^{m-1}\mathbb{Z}$, which is fixed by induction,
 596 so $p^{m-1}|n$, and in fact $p^{m-1} \neq n$ because $\phi(A)$ does not belong to the set of groups of the
 597 form $\mathbb{Z}^\ell \oplus \mathbb{Z}/p^{m-1}\mathbb{Z}$ which are all fixed by ϕ .

598 On the other hand, A doesn't preform any of the groups $\mathbb{Z}^\ell \oplus \mathbb{Z}/q\mathbb{Z}$ for $q \neq p$ prime,
 599 which are fixed, so $\phi(A)$ neither preform any of these groups, and $n = p^e$ for some $e \geq m$.

600 Now there are precisely $m + 1$ groups between A and \mathbb{Z}^{k+2} , namely all $\mathbb{Z}^{k+1} \oplus \mathbb{Z}/p^i\mathbb{Z}$
 601 for $i = 0, \dots, m$. This feature distinguishes A from $\mathbb{Z} \oplus \mathbb{Z}/p^e\mathbb{Z}$ for all $e \neq m$, and therefore
 602 A is fixed by ϕ . \square

603 **Lemma 3.12.** *Let ϕ be an order-preserving bijection of the infinite abelian groups, such
 604 that $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ for all primes p .*

605 *Then ϕ fixes all groups of the form $\mathbb{Z}^k \oplus C$ with C an abelian p -group.*

606 *Proof.* By Lemma 3.10, we have $\phi(\mathbb{Z}^k \oplus C) = \mathbb{Z}^k \oplus C'$ for a finite group C' with the
 607 same number of factors in a minimal decomposition as a product of cyclic groups.

608 Write $C = \bigoplus_{i=1}^r \mathbb{Z}/p^{e_i}\mathbb{Z}$, with $1 \leq e_1 \leq e_2 \leq \dots \leq e_r$. We proceed by induction on
 609 r , the case $r = 1$ being covered by Lemma 3.11.

610 Write $A = \mathbb{Z}^k \oplus C$. Since, when ℓ is large, $A \overset{\sim}{\sim} \mathbb{Z}^\ell \oplus \mathbb{Z}/q\mathbb{Z}$ with q prime if and only
 611 if $q = p$, we find that C' is a p -group of the form $\bigoplus_{i=1}^r \mathbb{Z}/p^{f_i}\mathbb{Z}$, with $1 \leq f_1 \leq \dots \leq f_r$.

612 Consider $B = \mathbb{Z}^{k+1} \oplus \bigoplus_{i=1}^{k-1} \mathbb{Z}/p^{e_i}\mathbb{Z}$, which is fixed by induction. We have $A \overset{\sim}{\sim} B$,
 613 so $\phi(A) \overset{\sim}{\sim} B$ and therefore $f_1 = e_1, \dots, f_{r-1} = e_{r-1}, f_r \geq e_r$ by Proposition 3.3. It
 614 remains to prove $f_r = e_r$.

615 Again by induction, the group $\mathbb{Z} \oplus B$ is fixed by ϕ . There are $e_r + 1$ groups between
 616 A and $\mathbb{Z} \oplus B$, namely $B \oplus \mathbb{Z}/p^e\mathbb{Z}$ for $e = 0, \dots, e_r$. This distinguishes A among all
 617 $\mathbb{Z}^k \oplus \bigoplus_{i=1}^{r-1} \mathbb{Z}/p^{e_i}\mathbb{Z} \oplus \mathbb{Z}/p^{f_r}\mathbb{Z}$ with $f_r \geq e_r$. \square

618 We are ready to finish the proof of Proposition 3.7. Consider again ϕ fixing all $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$
 619 for p prime, and an abelian group $A = \mathbb{Z}^k \oplus C$ with C finite; let us show that the torsion
 620 of $\phi(A)$ is isomorphic to C .

621 First, by Lemma 3.11, we have $\phi(A) = \mathbb{Z}^k \oplus C'$ for a finite group C' . Observe that,
 622 for ℓ large and D a p -group, A preforms $\mathbb{Z}^\ell \oplus D$ if and only if D is a subgroup of C . By
 623 Lemma 3.12, this group $\mathbb{Z}^\ell \oplus D$ is fixed by ϕ , so C and C' have the same p -subgroups.
 624 Since every abelian group is the product of its p -Sylow subgroups, it follows that C and
 625 C' are isomorphic. \square

626 **3.1. Virtually abelian groups.** There are countably many components of virtually abelian
 627 groups, as we now show:

628 **Example 3.13.** *Let $N_{2,2}$ be the group with presentation*

$$N_{2,2} = \langle a, b \mid c = [a, b] \text{ central} \rangle,$$

629 *and every $n \in \mathbb{N}$, let G_n be the virtually abelian group*

$$N_{2,2,n} = N_{2,2}/\langle c^n \rangle = \langle a, b \mid [a, b]^n, [a, b] \text{ central} \rangle.$$

630 *Then every $N_{2,2,n}$ is virtually \mathbb{Z}^2 , but if $m \neq n$ then $N_{2,2,n}$ and $N_{2,2,m}$ belong to different
 631 components of \mathcal{G}/\cong .*

632 *Proof.* Without loss of generality, assume $m < n$, and let H belong to the component
 633 of $N_{2,2,m}$; so there is a sequence $N_{2,2,m} = H_0, H_1, \dots, H_\ell = H$ with $H_i \overset{\sim}{\sim} H_{i-1}$ or

634 $H_{i-1} \xrightarrow{\sim} H_i$ for all $i = 1, \dots, \ell$. By Lemma 2.8(1,2), every H_i is finitely presented and
 635 satisfies the identity $[x, y]^m$. However, $N_{2,2,n}$ does not satisfy this identity. \square

636 **Remark 3.14.** If p is prime, then the set of groups limit greater than $N_{2,2,p}$ is precisely
 637 $\{N_{2,2,p} \times \mathbb{Z}^\ell : \ell \in \mathbb{N}\}$.

638 *Proof.* Elements of $N_{2,2,p}$ may uniquely be written in the form $a^x b^y c^z$ for some $x, y \in \mathbb{Z}$
 639 and $z \in \{0, \dots, p-1\}$. Consider a sequence of generating sets S_1, S_2, \dots of same cardi-
 640 nality k . Clearly, if each S_n is changed by a bounded number of Nielsen transformations,
 641 then without loss of generality one may assume (up to taking a subsequence) that the same
 642 transformations are applied to all S_n , and therefore the limit does not change.

643 Using at most pk transformations, the set S_n , whose elements we write as $\{s_{n,1}, \dots, s_{n,k}\}$,
 644 can be transformed in such a manner that two elements $s_{n,1}, s_{n,2}$ generate $N_{2,2,p}$ while the
 645 other $s_{n,3}, \dots, s_{n,k}$ are of the form $a^x b^y c^z$ with $p|x$ and $p|y$, and therefore belong to the
 646 centre of $N_{2,2,p}$. Some of these elements will belong to $\langle s_1, s_2 \rangle$ in the limit, and others
 647 will generate extra abelian factors. \square

648 4. NILPOTENT GROUPS

649 Given a group G , we denote its lower central series by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) =$
 650 $[G, \gamma_i(G)]$ for all $i \geq 1$. By $N_{s,k} = \mathbb{F}_k / \gamma_{s+1}(\mathbb{F}_k)$ we denote the free nilpotent group of
 651 class s on k generators.

652 We study in this section the structure of connected components of nilpotent groups; our
 653 main result is that, if $G/\text{Torsion}(G)$ generates the same variety as G , then the connected
 654 component of G is determined by the variety that it generates and conversely.

655 **4.1. Free groups and subgroups in nilpotent varieties.** Following [44, Definition 17.12],
 656 a group G is said to be *discriminating* if, given any finite set \mathcal{W} of identities that do not hold
 657 in G (i.e., for every $w \in \mathcal{W}$ there are $g_1, g_2, \dots \in G$ with $w(g_1, \dots) \neq 1$), all identities
 658 can be falsified simultaneously (i.e. there are $g_1, g_2, \dots \in G$ such that $w(g_1, \dots) \neq 1$ for
 659 all $w \in \mathcal{W}$). We will say G is *discriminating on k generators* if, given any finite set \mathcal{W} of
 660 identities in k letters that do not hold in G (i.e., for every $w \in \mathcal{W}$ there are $g_1, \dots, g_k \in G$
 661 with $w(g_1, \dots, g_k) \neq 1$), all identities can be falsified simultaneously on a generating set
 662 (i.e. there exists a generating set $\{g_1, \dots, g_k\}$ of G such that $w(g_1, \dots, g_k) \neq 1$ for all
 663 $w \in \mathcal{W}$).

664 Baumslag, Neumann, Neumann, and Neumann show in [13, Corollary 2.17] that finitely
 665 generated torsion-free nilpotent groups are discriminating; see also [44, Theorem 17.9]. If
 666 G is a nilpotent group with torsion, the matter is more delicate: Baumslag and Neumanns
 667 prove in the same place that G is discriminating if and only if G and $G/\text{Torsion}(G)$ gen-
 668 erate the same variety.

669 **Lemma 4.1.** *Let G be a discriminating group, and let \mathcal{V} be the variety generated by G .
 670 Let $\mathbb{V}_k := \mathbb{F}_k / \mathcal{V}(\mathbb{F}_k)$ be the free group on k generators in \mathcal{V} . Then for every $k \in \mathbb{N}$ there
 671 exists a group H that is preformed by G and contains \mathbb{V}_k as a subgroup.*

672 *If furthermore G is discriminating on k generators, then G preforms \mathbb{V}_k .*

673 *Proof.* Consider first a finite set of words $\mathcal{W} \subset \mathbb{F}_k$ that are not identities of \mathbb{V}_k , that
 674 is $w(s_1, \dots, s_k) \neq 1$ in \mathbb{V}_k for all $w \in \mathcal{W}$, with $\{s_1, \dots, s_k\}$ a free generating set
 675 for \mathbb{V}_k . Observe that, for each $w \in \mathcal{W}$, there exist elements $g_{w,1}, \dots, g_{w,k} \in G$ with
 676 $w(g_{w,1}, \dots, g_{w,k}) \neq 1$; otherwise, w would be an identity in G and therefore would
 677 vanish on \mathbb{V}_k . Since G is discriminating, there exist $g_{\mathcal{W},1}, \dots, g_{\mathcal{W},k} \in G$ such that
 678 $w(g_{\mathcal{W},1}, \dots, g_{\mathcal{W},k}) \neq 1$ for all $w \in \mathcal{W}$.

679 We apply this with \mathcal{W} the set of words of length at most R in \mathbb{F}_k that are not identities
 680 in \mathbb{V}_k , and denote the resulting $g_{\mathcal{W},1}, \dots, g_{\mathcal{W},k}$ by $g_{R,1}, \dots, g_{R,k}$.

681 Let S be a finite generating set for G , and put $S_R = S \sqcup \{g_{R,1}, \dots, g_{R,k}\}$. Choose
 682 an accumulation point (H, T) of the sequence (G, S_R) in the space \mathcal{G} of marked groups.
 683 Then H contains \mathbb{V}_k as the subgroup generated by the limit of $\{g_{R,1}, \dots, g_{R,k}\}$.

684 If G is discriminating on k generators, then we can take $S = \emptyset$ in the previous para-
 685 graph, to see that H is isomorphic to the relatively free group \mathbb{V}_k . \square

686 For a real constant C , let us say that the sequence of positive real numbers x_1, x_2, \dots, x_s
 687 grows at speed C if $x_1 \geq C$ and $x_{i+1} \geq x_i^C$ for $i = 1, \dots, s-1$. Similarly, an unordered
 688 set $\{x_1, \dots, x_s\}$ grows at speed C if it admits an ordering that grows at speed C .

689 **Lemma 4.2.** *Suppose that f_1, \dots, f_t are nonzero polynomials in s variables with real*
 690 *coefficients. Then there exists C such that $f_i(x_1, \dots, x_s) \neq 0$ for all $i = 1, \dots, t$ whenever*
 691 *(x_1, \dots, x_s) grows at speed C .*

692 *Proof.* It suffices to prove the statement for a single polynomial f . Let $x_1^{e_1} \dots x_s^{e_s}$ be the
 693 lexicographically largest monomial in f ; namely, e_s is maximal among all monomials in
 694 f ; then e_{s-1} is maximal among monomials of degree e_s in x_s ; etc. Then this monomial
 695 dominates f as (x_1, \dots, x_s) grows faster and faster. \square

696 **Lemma 4.3.** *Consider $d \geq 1$. Then for all $e \geq d+1$ and all $C > 0$ there exists a set*
 697 *of numbers $\{x_{1,1}, x_{1,2}, \dots, x_{1,d}, x_{2,1}, \dots, x_{e,1}, x_{e,d}\}$ growing at speed C and such that*
 698 *$\{(x_{1,1}, \dots, x_{1,d}), \dots, (x_{e,1}, \dots, x_{e,d})\}$ is a generating set for \mathbb{Z}^d .*

699 *Proof.* It suffices to prove the statement for $e = d+1$. We start by proving the following
 700 claim by induction on $n = 1, \dots, d$: there exists an $n \times n$ integer matrix $(x_{i,j})$ whose
 701 coefficients grow at speed C , and such that for every $k = 1, \dots, n$ the determinant of the
 702 upper left corner $(x_{i,j} : 1 \leq i, j \leq k)$ is a prime number p_k , with all primes p_1, \dots, p_n
 703 distinct.

704 The induction starts by setting $x_{1,1} = p_1$ for some prime number $p_1 > C$.

705 Assume then that an $(n-1) \times (n-1)$ matrix $A_{n-1} = (x_{i,j})$ has been constructed,
 706 with entries growing at speed C and determinant a prime number p_{n-1} .

707 First, an n th row $(x_{n,1}, \dots, x_{n,n-1})$ may be added to A_{n-1} in such a manner that the
 708 entries still grow at speed C , and the determinant d_n of $A'_{n-1} = (x_{i,j} : i \neq n-1)$ is
 709 coprime to p_{n-1} . Indeed the coefficients $x_{n,1}, \dots, x_{n,n-2}$ may be chosen arbitrarily as
 710 long as they grow fast enough. Then increasing $x_{n,n-1}$ increases the determinant of A'_{n-1}
 711 by p_{n-2} which is coprime to p_{n-1} ; and sufficiently increasing this coefficient makes the
 712 augmented matrix $A''_{n-1} = (x_{i,j} : i \leq n)$ still growing at speed C .

713 Then an n th column may be added to A''_{n-1} as follows. Start by choosing $x_{1,n}, \dots, x_{n-2,n}$
 714 arbitrarily as long as they grow fast enough, without fixing $x_{n-1,n}$ and $x_{n,n}$ yet. Call A_n
 715 the resulting matrix. Then increasing $x_{n-1,n}$ decreases the determinant of A_n by d_n , while
 716 increasing $x_{n,n}$ increases the determinant of A_n by p_{n-1} . Since d_n and p_{n-1} are coprime,
 717 there exist choices of $x_{n-1,n}$ and $x_{n,n}$ such that A_n has determinant 1; and the entries of
 718 A_n grow at speed C , except perhaps for $x_{n,n}$.

719 Now, by Dirichlet's theorem, there exists arbitrarily large primes p_n that are $\equiv 1$
 720 (mod p_{n-1}). For such a prime $p_n = 1 + ap_{n-1}$, add a to the entry $x_{n,n}$ yielding a
 721 matrix A_n of determinant p_n . Choosing a large enough makes the coefficients of A_n grow
 722 at speed C .

723 To prove the lemma, consider a $d \times d$ matrix A with integer entries growing at speed C
 724 and determinant p . Its rows generate a subgroup of \mathbb{Z}^d of prime index, and a single extra

generator, with fast growing entries that are coprime to p , gives the desired generating set. \square

We are ready to sharpen [13, Corollary 2.17], claiming that torsion-free nilpotent groups are discriminating:

Lemma 4.4. *Let G be a torsion-free k -generated nilpotent group. Then, for each $N > k$, the group G is discriminating on N generators.*

Proof. We start by considering more generally poly- \mathbb{Z} groups, namely groups G admitting a sequence of subgroups $G = G_1 \supset G_2 \supset \cdots \supset G_{\ell+1} = 1$ such that $G_i/G_{i+1} \cong \mathbb{Z}$ for all i .

If G is torsion-free nilpotent and (Z_i) denotes its ascending central series (defined inductively by $Z_0 = 1$ and $Z_{i+1}/Z_i = Z(G/Z_i)$), then each Z_{i+1}/Z_i is free abelian, so the ascending central series can be refined to a series in which successive quotients are \mathbb{Z} .

Choose for all $i = 1, \dots, \ell$ a generator of G_i/G_{i+1} , and lift to an element $u_i \in G_i$. Then every $g \in G$ may uniquely be written in the form $g = u_1^{\xi_1} \cdots u_{\ell}^{\xi_{\ell}}$, and the integers ξ_1, \dots, ξ_{ℓ} determine the element g , which we write u^{ξ} . Philip Hall proved in [34, Theorem 6.5] that products and inverses are given by polynomials, in the sense that if $u^{\xi} u^{\eta} = u^{\zeta}$ and $(u^{\xi})^{-1} = u^{\chi}$, then ζ_i and χ_i are polynomials in $\{\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}\}$ and $\{\xi_1, \dots, \xi_{\ell}\}$ respectively. In particular, every identity $w \in \mathcal{W}$, in N variables, is a polynomial in the exponents $\xi_{1,1}, \dots, \xi_{\ell,N}$ of its arguments x_1, \dots, x_N written as $u^{\xi_1}, \dots, u^{\xi_N}$.

By Lemma 4.3, there exist sequences with arbitrarily fast growth that generate the abelianization of G ; and by Lemma 4.2 the identities in \mathcal{W} will not vanish on these generators, if their growth is fast enough. Finally, since G is nilpotent, a sequence of elements generates G if and only if it generates its abelianization. \square

Lemma 4.5. *Let G be a finitely generated nilpotent group such that G and $G/\text{Torsion}(G)$ generate the same variety. Then G preforms a torsion-free nilpotent group.*

Proof. Infinite, finitely generated nilpotent groups have infinite abelianization; we apply Lemma 2.19 to G and the variety \mathcal{V} of abelian groups. Since every infinite abelian group preforms a free abelian group, we assume without loss of generality that G has torsion-free abelianization.

Assume that G is k -generated, and consider $N > k$ and $R > 0$. Consider the set $\mathcal{W}(R)$ of all words w of length at most R in N variables such that, for some $g_1, \dots, g_N \in G$, the evaluation $w(g_1, \dots, g_N)$ is a non-trivial torsion element in G . In particular, such w are not identities in G . Since G and $G/\text{Torsion}(G)$ generate the same variety, none of these words is an identity in $G/\text{Torsion}(G)$. Since $G/\text{Torsion}(G)$ is a torsion-free nilpotent group, Lemma 4.4 implies that $\mathcal{W}(R)$ is discriminated by an N -element generating set of $G/\text{Torsion}(G)$, which we denote by S'_R . Let S_R denote a preimage in G of S'_R . Since the abelianization of G is torsion-free, it is isomorphic (under the natural quotient map) to the abelianization of $G/\text{Torsion}(G)$. Therefore, S_R generates the abelianization of G , so generates G .

Let (H, T) be an accumulation point of the sequence (G, S_R) in the space \mathcal{G} of marked groups. Observe that H is torsion-free. Indeed, by Lemma 2.8(3) the torsion of H imbeds in that of G ; and if a is a torsion element of G , then for all R large enough there are words $w \in \mathcal{W}(R)$ that assume the value a . By construction of S_R , the value a is not taken by a word of length $\leq R$ in S_R , so a does not have a limit in H . \square

Proposition 4.6. *Let G be a k -generated nilpotent group, and assume that G and $G/\text{Torsion}(G)$ generate the same variety, \mathcal{V} .*

771 *Then, for every $N > k$, the group G preforms \mathbb{V}_N .*

772 *Consequently, the connected component of G for the relation $\overset{\sim}{\rightsquigarrow}$ has diameter 2.*

773 *Proof.* By Lemma 4.5, we may assume that G is torsion-free nilpotent. By Lemma 4.4, the
774 group G is discriminating on N generators. By Lemma 4.1, the group G precedes \mathbb{V}_N . \square

775 **Remark 4.7.** The assumption that G is torsion-free is essential for the first claim of the
776 proposition above. Consider indeed the variety of nilpotent groups of nilpotent class 2 in
777 which every commutator is of order p . This variety is generated, e.g., by the group $N_{2,2,p}$
778 from Example 3.13. However, there does not even exist any group preformed by G and
779 containing \mathbb{V}_3 as a subgroup, because the torsion \mathbb{V}_3 is larger than the torsion in $N_{2,2,p}$.

780 **Remark 4.8.** Let \mathcal{V} be a nilpotent variety. Then, if $\mathbb{V}_m \overset{\sim}{\rightsquigarrow} \mathbb{V}_n$, then $m \leq n$.

781 *Proof.* Since \mathbb{V}_n is finitely presented, \mathbb{V}_m is a quotient of \mathbb{V}_n . The abelianization of \mathbb{V}_n
782 is n -generated, so the abelianization of any quotient of \mathbb{V}_n is also n -generated, so $m \leq$
783 n . \square

784 Proposition 4.6 has the following

785 **Corollary 4.9.** *Consider a nilpotent variety \mathcal{V} generated by a group G such that $G/\text{Torsion}(G)$
786 also generates \mathcal{V} . Let c be the nilpotency class of G .*

787 *For $m, n > c$, we have $\mathbb{V}_m \overset{\sim}{\rightsquigarrow} \mathbb{V}_n$ if and only if $m \leq n$.*

788 *Proof.* It is known from [44, Theorem 35.11] that \mathbb{V}_m generates \mathcal{V} as soon as $m \geq c$. \square

789 **Remark 4.10.** Consider a nilpotent variety \mathcal{V} generated by a torsion-free nilpotent group.
790 For small m, n , the free groups \mathbb{V}_m and \mathbb{V}_n need not belong to the same component. For
791 example, if \mathcal{V} the variety of nilpotent groups of class 5, then \mathbb{V}_2 does not generate \mathcal{V} , since
792 it is metabelian but \mathbb{V}_3 is not. See [44, 35.33] for details.

793 **4.2. When generators of a variety lie in different components.** We will see that, if G
794 and $G/\text{Torsion}(G)$ lie in different varieties, then the variety of G contains infinitely many
795 connected components under $\overset{\sim}{\rightsquigarrow}$.

796 **Lemma 4.11.** *Let G be a nilpotent group such that G and $G/\text{Torsion}(G)$ generate differ-
797 ent varieties. There exists a variety \mathcal{V} such that the verbal subgroup $\mathcal{V}(G)$ is non-trivial
798 and finite.*

799 *Proof.* First recall that torsion elements of a nilpotent group G form a finite subgroup of
800 G . Since G and $G/\text{Torsion}(G)$ generate different varieties, there exists an identity w of
801 $G/\text{Torsion}(G)$ that is not an identity in G . Set $\mathcal{V} = \{w\}$; then $\mathcal{V}(G)$ is non-trivial and is
802 contained in the torsion of G , hence finite. \square

803 **Corollary 4.12.** *Let G be a nilpotent group and let \mathcal{V} be the variety that it generates. The
804 connected component of G coincides with the set of groups generating \mathcal{V} if and only if
805 $G/\text{Torsion}(G)$ generates \mathcal{V} . If this is not the case, the set of groups generating \mathcal{V} consists
806 of infinitely many connected components for the relation $\overset{\sim}{\rightsquigarrow}$.*

807 *Proof.* If $G/\text{Torsion}(G)$ generates \mathcal{V} , the corollary follows from Proposition 4.6. Assume
808 now that $G/\text{Torsion}(G)$ does not generate \mathcal{V} . Then by Lemma 4.11 there exists a variety
809 \mathcal{W} such that the verbal subgroup $\mathcal{W}(G)$ is non-trivial and finite. Observe that a verbal
810 subgroup of a direct product is the product of its verbal subgroups. Therefore, for all
811 $n \in \mathbb{N}$, the verbal subgroups $\mathcal{W}(\times_n G)$ are non-isomorphic. By Lemma 2.25, all the
812 groups $\times_n G$ lie in distinct connected components. However, they all generate \mathcal{V} . \square

813 **4.3. Examples and illustrations.** In the variety of abelian groups, the following is true:
 814 if G is a quotient of H and the torsion of H imbeds in the torsion of G under the quotient
 815 map, then $G \lesssim H$. This is not true anymore among nilpotent groups.

816 **Example 4.13.** Consider the groups $G = N_{2,2}$ and $H = N_{2,2} \times N_{2,2}$, see Example 3.13.
 817 Then both G and H are torsion-free, and G is a quotient of H . However, G doesn't preform
 818 H .

819 *Proof.* Consider the following universal statement:

$$\forall a, b, c, z (([a, b] = 1 \wedge [a, c] = 1 \wedge [b, c] \neq 1) \Rightarrow [a, z] = 1).$$

820 It states that if a commutes with two non-commuting elements b and c , then a is central.

821 This property does not hold in H : take a, z the generators of the first $N_{2,2}$ and b, c the
 822 generators of the second one.

823 On the other hand, in $N_{2,2}$, this property holds. Indeed if $[a, b] = 1$ then the image of
 824 $\{a, b\}$ in $N_{2,2}/Z(N_{2,2}) \cong \mathbb{Z}^2$ lies in a cyclic subgroup; Similarly the image of $\{a, c\}$ lies in
 825 a cyclic subgroup; so either a is central or the image of $\{b, c\}$ lies in a cyclic subgroup. \square

826 **Example 4.14.** As soon as the nilpotency class is allowed to grow beyond 4, there exist
 827 nilpotent varieties whose free groups are not virtually free nilpotent. For example, consider
 828 the group $G = \mathbb{F}_3 / \langle \mathbb{F}_3'', \gamma_5(\mathbb{F}_3) \rangle$. This group is nilpotent of class 4, and is an iterated
 829 central extension of 29 copies of \mathbb{Z} . The 3-generated free nilpotent groups of class 3 and
 830 4 have respectively 14 and 32 cyclic factors, so G is not commensurable to either. This is
 831 easily seen in the (Malcev) Lie algebra associated with these groups.

832 **Lemma 4.15.** Let G be a non-virtually abelian nilpotent group. Then the connected com-
 833 ponent of G is not isomorphic, as partially ordered set, to the component of abelian groups.

834 *Proof.* In the component of abelian groups, the following holds: for any A there exists B
 835 with $A \lesssim B$ and such that the set of groups that are preformed by B is linearly ordered.
 836 We claim that the connected component of G does not have this property.

837 More precisely, for any non-virtually abelian nilpotent G , we construct incomparable
 838 groups H_1, H_2 that are both preformed by G .

839 Since G is not virtually abelian, $[G, G]$ is infinite. Then both G and $[G, G]$ have in-
 840 finite abelianization, so that G maps onto $N_{2,2}$, the free nilpotent group of class 2 on 2
 841 generators. Since $N_{2,2} \lesssim N_{2,k}$ for all $k \geq 2$, there exists by Lemma 2.18 a group H_1
 842 such that $\gamma_2(H_1)/\gamma_3(H_1)$ has arbitrarily large rank, in particular rank larger than that of
 843 $\gamma_2(G)/\gamma_3(G)$. Set then $H_2 = G \times \mathbb{Z}^d$ for d larger than the rank of $H_1/\gamma_2(H_1)$. Then H_1
 844 is not a quotient of H_2 , because $\gamma_2(H_1)/\gamma_3(H_1)$ is not a quotient of $\gamma_2(H_2)/\gamma_3(H_2)$; and
 845 H_2 is not a quotient of H_1 , because $H_2/\gamma_2(H_2)$ is not a quotient of $H_1/\gamma_2(H_1)$. \square

846 5. IMBEDDABILITY OF ORDERS. SOLVABLE GROUPS

847 We characterize the preorders (transitive, reflexive relations) that can be imbedded in
 848 the preorder of groups up to isomorphism, under the relation \lesssim . We show in this manner
 849 that \lesssim has a rich structure, even when restricted to solvable groups of class 3.

850 In this section, we view \lesssim as a preorder on \mathcal{G} , defined by $(G, S) \lesssim (H, T)$ if and only
 851 if $G \lesssim H$. For X a set, we denote by $\mathcal{P}(X)$ the family of subsets of X .

852 **Proposition 5.1.** Let \mathcal{B} be a countably infinite set, and let \mathcal{X} have the cardinality of the
 853 continuum. Put on $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$ the preorder

$$(X, c) \lesssim (Y, c') \text{ if and only if } X \supseteq Y.$$

854 Then the preorders (\mathcal{G}, \lesssim) and $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$ imbed into each other.

855 We note that $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$ is the relation obtained by the partial order on subsets of
 856 \mathcal{B} by inclusion; its equivalence classes (strongly connected components) have the cardinality of the continuum. We also remark that $(\mathcal{P}(\mathcal{B}), \subseteq)$ is isomorphic to $(\mathcal{P}(\mathcal{B}), \supseteq)$, via
 857 the map $X \mapsto \mathcal{B} \setminus X$.
 858

859 **Corollary 5.2.** *A preorder imbeds in $(\mathcal{G}/\cong, \lesssim)$ if and only if it imbeds in $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$.
 860 In particular, a partial order imbeds in $(\mathcal{G}/\cong, \lesssim)$ if and only if it is realizable by subsets
 861 of a countable set under inclusion.*

862 *Proof.* Proposition 5.1 yields imbeddings between \mathcal{G} and $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$. We therefore have
 863 an imbedding of \mathcal{G}/\cong into $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$.

864 Conversely, isomorphism classes of groups in \mathcal{G} are countable, because there are countably many homomorphisms between finitely generated groups. On the other hand, equivalence classes in $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$ are uncountable; so there exists an imbedding $\mathcal{P}(\mathcal{B}) \times \mathcal{X} \rightarrow$
 865 $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$, which is the identity on its first argument, and such that its image imbeds in
 866 \mathcal{G}/\cong . \square

869 *Proof of Proposition 5.1, \hookrightarrow .* Consider first the space \mathcal{G} of marked groups. For every
 870 $k, R \in \mathbb{N}$, there are finitely many possibilities for the marked graphs $B(1, R)$ of degree
 871 $\leq k$ that may appear in the Cayley graphs of these groups; letting k, R range over \mathbb{N} , we
 872 obtain a countable collection \mathcal{B} of finite graphs. Now to each $(G, S') \in \mathcal{G}$ we associate the
 873 subset \mathcal{O}_G of \mathcal{B} consisting of all marked balls that may appear in Cayley graphs $\mathcal{C}(G, S)$,
 874 as we let S range over generating sets of G . Clearly, $G \lesssim H$ if and only if $\mathcal{O}_H \subseteq \mathcal{O}_G$.

875 We deduce that (\mathcal{G}, \lesssim) imbeds in $(\mathcal{P}(\mathcal{B}), \subseteq)$. We can make this map injective by
 876 taking $\mathcal{X} = \mathcal{P}(\mathbb{F})$, and mapping (G, S) to $(\mathcal{O}_G, \ker(\mathbb{F} \rightarrow G))$, for the natural map $\mathbb{F} \rightarrow G$
 877 presenting G . \square

878 To construct the imbedding in the other direction, we begin by a general construction.
 879 Let P be a group. Consider first the free nilpotent group $N_{2,P}$ of class 2 on a generating
 880 set indexed by P . Denote its generators by a_p for $p \in P$, and for $p, q \in P$ write $c_{p,q} :=$
 881 $[a_p, a_q]$. We have $c_{p,p} = 0$, and $c_{p,q} = -c_{q,p}$ for all $p, q \in P$. Define then $\overline{N}_{2,P}$ as the
 882 quotient of $N_{2,P}$ by the relations $c_{p,q} = c_{pr,qr}$ for all $p, q, r \in P$. Finally let $H(P)$ be the
 883 semidirect product $P \ltimes \overline{N}_{2,P}$, for the action $a_p \cdot q := a_{pq}$. The centre of $H(P)$ is generated
 884 by the images of the $c_{p,q}$. Let $P_+ \subseteq P \setminus \{1\}$ contain precisely one element out of each pair
 885 $\{p, p^{-1}\}$; then $\{c_{1,p}\}$ freely generates the centre of $H(P)$. If S be a generating set for P ,
 886 then $S \cup \{a_1\}$ generates $H(P)$.

887 The case $P = \mathbb{Z}$ is considered by Hall in [33, §3]; he introduced this group in order to
 888 construct 2^{\aleph_0} non-isomorphic solvable finitely generated groups (of solvability length 3).

889 In this proof, we take $P = \mathbb{Z}^2$, and for convenience $(\mathbb{Z}^2)_+ = \{(m, n) \in \mathbb{Z}^2 : m >$
 890 $0 \text{ or } m = 0 < n\}$. We abbreviate $H(\mathbb{Z}^2)$ as H , generated by $\{x, y, a\}$ with $\{x, y\}$ the
 891 standard generators of \mathbb{Z}^2 and $a = a_{(0,0)}$.

892 A prime colouring is a function $\phi: (\mathbb{Z}^2)_+ \rightarrow \{1\} \cup \{\text{primes}\}$; it extends to a function
 893 still written $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by $\phi(-z) = -\phi(z)$ and $\phi(0) = 0$. Given a prime colouring ϕ , we
 894 define the standard quotient H_ϕ of H as the quotient of H by all the relations $c_{1,z}^{\phi(z)} = 1$,
 895 as z ranges over $(\mathbb{Z}^2)_+$. Clearly,

896 **Lemma 5.3.** *A standard central quotient H_ϕ contains an element of order p if and only if
 897 there exists $z \in (\mathbb{Z}^2)_+$ such that $\phi(z) = p$.*

898 If $H_\phi \overset{\sim}{\hookrightarrow} H_\psi$, then the set of primes in $\psi(\mathbb{Z})$ is contained in the set of primes in $\phi(\mathbb{Z})$.
 899 \square

900 Let I be a set of primes. A prime colouring ϕ is *I-universal* if its values lie in I and
 901 it contains every finite I -colouring, in the following sense: for every $R \in \mathbb{N}$ and every
 902 function $\theta : \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$, there exists $M \in \mathbf{SL}_2(\mathbb{Z})$ such that
 903 $\theta(z) = \phi(M(z))$ for all $z \in \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+$.

904 **Lemma 5.4.** *For every set I of primes of cardinality ≥ 2 , there exist a continuum of*
 905 *I -universal colourings.*

906 *Proof.* One enumerates all finite I -colourings, and constructs ϕ step by step. At each step,
 907 only finitely many values of ϕ have been specified, say within the box $\{-S, \dots, S\}^2$, and
 908 we want to extend ϕ using the partial colouring $\theta : \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$. A
 909 large enough $M \in \mathbf{SL}_2(\mathbb{Z})$ can be found such that $M(\{-R, \dots, R\}^2 \cap \{-S, \dots, S\}^2) =$
 910 $\{(0, 0)\}$, for example $M = \begin{pmatrix} (S+1)(S+R+1)+1 & S+1 \\ S+R+1 & 1 \end{pmatrix}$. Extend ϕ by setting $\phi(M(z)) = \theta(z)$
 911 for all $z \in \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+$. Once this is done for all $R \in \mathbb{N}$, set finally $\phi(z) = 1$ at
 912 unspecified values in $(\mathbb{Z}^2)_+$.

913 To obtain a continuum of different I -universal colourings, note that countably many
 914 matrices M_0, M_1, \dots were used in the construction, and the only condition was that they
 915 had to be sufficiently far away from the identity. Fix a finite-index subgroup $\Gamma \subset \mathbf{SL}_2(\mathbb{Z})$.
 916 Then, given a subset $C \subseteq \mathbb{N}$, one may choose the matrices M_i as above, and additionally
 917 such that $M_i \in \Gamma \Leftrightarrow i \in C$. This encodes C into the constructed colouring. \square

918 *Proof of Proposition 5.1, \Leftarrow .* We are ready to imbed $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$ into \mathcal{G} . Without loss of
 919 generality, we may assume that \mathcal{B} is the set of primes ≥ 10 .

920 Given $X \subseteq \mathcal{B}$, consider $I = \{2, 3\} \cup X$. By Lemma 5.4, there exist continuously
 921 many I -universal prime colourings $\phi_{I,C}$, parameterized by $C \subseteq \mathbb{N}$. Let $H_{X,C}$ be the
 922 central quotient $H_{\phi_{X \cup \{2,3\}, C}}$, and note that the $(H_{X,C}, \{x, y, a\})$ are distinct points of \mathcal{G}
 923 for distinct (X, C) . We have therefore defined an imbedding $\mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{G}$.

924 On the one hand, if $H_{X,C} \overset{\sim}{\hookrightarrow} H_{Y,C'}$, then $X \supseteq Y$ by Lemma 5.3. On the other hand, if
 925 $X \supseteq Y \subseteq \mathcal{B}$ and $C, C' \subseteq \mathbb{N}$, then consider the prime colourings ϕ, ψ with $H_{X,C} = H_\phi$
 926 and $H_{Y,C'} = H_\psi$, and choose $T = \{x, y, a\}$ as generating set of H_ψ . Consider an arbitrary
 927 $R \in \mathbb{N}$. Then the restriction of ψ to $\{-R, \dots, R\}^2$ is a finite $(\{2, 3\} \cup Y)$ -colouring, and
 928 therefore a finite $(\{2, 3\} \cup X)$ -colouring; so there exists $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ such that
 929 ψ and $\phi \circ M$ agree on $\{-R, \dots, R\}^2$. Consider the generating set $S = \{x^a y^b, x^c y^d, a\}$ of
 930 H_ϕ ; then the Cayley graphs $\mathcal{C}(H_\psi, T)$ and $\mathcal{C}(H_\phi, S)$ agree on a ball of radius R . \square

931 **Remark 5.5.** By Lemma 2.8(3), if $A \overset{\sim}{\hookrightarrow} B$ and F is a finite subgroup of B , then F imbeds
 932 in A . In general, if F is a torsion subgroup of B , this need not be true. There exist finitely
 933 generated solvable groups $A \overset{\sim}{\hookrightarrow} B$, such that B contains the divisible group \mathbb{Q}/\mathbb{Z} , while A
 934 does not contain any divisible elements.

935 *Proof.* We modify the proof of Proposition 5.1. Before, we enumerated finite I -colourings
 936 $\theta : \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$ and imposed the relations $c_{1,M(z)}^{\theta(z)} = 1$, for appro-
 937 priate $M \in \mathbf{SL}_2(\mathbb{Z})$. Now, we enumerate $(\mathbb{Z}^2)_+$ as $\{p_1, p_2, \dots\}$, and we impose relations
 938 on H step-by-step. At each step, only finitely many of the $c_{1,z}$ will have been affected by
 939 the relations; we call the corresponding $z \in \mathbb{Z}^2$ *bound*.

940 For each $N = 1, 2, \dots$, we find $M \in \mathbf{SL}_2(\mathbb{Z})$ such that $M(\{p_1, \dots, p_N\})$ is disjoint
 941 from all bound $z \in \mathbb{Z}^2$. We impose the relations $c_{1,M(p_1)} = 1$ and $c_{1,M(p_i)}^i = c_{1,M(p_{i-1})}$
 942 for all $i = 2, \dots, N$. Finally, we set $c_{1,z} = 1$ for all unbound $z \in \mathbb{Z}^2$.

943 We call the resulting central quotient G , and note that it is solvable, and that its torsion
 944 is the subgroup generated by the $c_{1,z}$; this group is a direct sum of cyclic groups, and in
 945 particular is not divisible.

946 On the other hand, let (H, T) be the limit of (G, S_M) in the space \mathcal{G} of marked groups,
 947 along the generating sets $S_M = \{x^a y^b, x^c y^d, a\}$ corresponding to the matrices $M =$
 948 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ used in the construction of G . Then H contains a copy of \mathbb{Q}/\mathbb{Z} , with
 949 the limit of $c_{1,M(p_i)}$ playing the role of $1/i!$. \square

950 6. THE CONNECTED COMPONENT OF FREE GROUPS

951 We concentrate, in this section, on those groups that either preform or are preformed by
 952 free groups. Both of these classes have already been thoroughly investigated; the first are
 953 known as “limit groups”, and the second as “groups without almost-identities”.

954 **6.1. Limit groups.** Groups that are preformed by free groups are known as “limit groups”.
 955 This section reviews some known facts about them; we refer to the recent expositions [14,
 956 39, 46].

957 Benjamin Baumslag considered residually free groups in [11]. An ω -residually free
 958 groups is a group G such that, for all n and all distinct $g_1, \dots, g_n \in G$, there exists a
 959 homomorphism $\pi: G \twoheadrightarrow \mathbb{F}$ to a free group such that all $\pi(g_1), \dots, \pi(g_n)$ are distinct.
 960 Baumslag proved in particular that G is ω -residually free if and only if it is both residually
 961 free and commutative-transitive (see Equation 1).

962 Remeslennikov proved in [50] that the following are equivalent for a residually free
 963 group: it is ω -residually free; it is universally free (namely has the same universal theory
 964 as a free group); it is commutative transitive (see Equation 1). All three statements are
 965 characterizations of non-abelian *limit groups*. The terminology was introduced by Sela,
 966 referring to limits of epimorphisms onto free groups.

967 Champetier and Guirardel show in [19] that G is a limit group if and only if it is a limit
 968 of subgroups of free groups. In other words, G is a non-abelian limit group if and only if
 969 $\mathbb{F}_2 \overset{\sim}{\hookrightarrow} G$.

970 Kharlampovich-Myasnikov [37, 38] and Sela [52] prove that limit groups are finitely
 971 presented.

972 **6.2. Groups groups with no almost-identities.** Groups that preform free groups will be
 973 shown to be “groups with no almost-identities”. We write $G \overset{\sim}{\hookrightarrow} \mathbb{F}$ if there exists $k \in \mathbb{N}$
 974 such that $G \overset{\sim}{\hookrightarrow} \mathbb{F}_k$; equivalently, $G \overset{\sim}{\hookrightarrow} \mathbb{F}_k$ for all k large enough.

975 We begin by some elementary observations and examples. We include the proofs for
 976 convenience of the reader.

977 **Lemma 6.1** (See [51] and [19, Example 2.4(d)]). *We have $\mathbb{F}_m \overset{\sim}{\hookrightarrow} \mathbb{F}_n$ if and only if $m \leq n$.
 978 More precisely, let $\{x_1, \dots, x_m\}$ be a basis of \mathbb{F}_m and let S_R be, for all $R \in \mathbb{N}$, a set
 979 of $n - m$ words of length at least $2R$ satisfying the $C'(1/6)$ small cancellation condition.
 980 Then $(\mathbb{F}_m, \{x_1, \dots, x_m\} \cup S_R)$ converges to $(\mathbb{F}_n, \text{basis})$ in \mathcal{G} .*

981 *Proof.* Consider $m \leq n$. Let $S = \{x_1, \dots, x_m\}$ be a basis of \mathbb{F}_m . Given $R > 0$, con-
 982 sider a set $S_R := \{w_1, \dots, w_{n-m}\}$ such that each word w_i has length larger than $2R$,
 983 and $\{w_1, \dots, w_{n-m}\}$ satisfies the $C'(1/6)$ small cancellation condition. The presentation
 984 $\langle x_1, \dots, x_m, y_1, \dots, y_{n-m} \mid y_1 w_1, \dots, y_{n-m} w_{n-m} \rangle$ then defines the free group \mathbb{F}_m , and
 985 also satisfies the $C'(1/6)$ small cancellation condition. By Greendlinger’s Lemma [25], the
 986 shortest relation in it has length larger than $2R$, so the ball of radius R in $\mathcal{C}(\mathbb{F}_m, \{x_1, \dots, x_m\} \cup S_R)$
 987 coincides with that in \mathbb{F}_n .

Conversely, if $\mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}_n$ then \mathbb{F}_m is a quotient of \mathbb{F}_n , by Lemma 2.4, so $m \leq n$. \square

Lemma 6.2 (See [51, Lemma 5.1]). *If G be an s -generated group which admits \mathbb{F}_m as a quotient, for some $m \geq 2$, then G preforms a free group on $m + s$ elements.*

Proof. Let $\{g_1, \dots, g_s\}$ generate G , and let g'_1, \dots, g'_s be the projections of the g_i to \mathbb{F}_m . Let also $h_1, \dots, h_m \in G$ project to a basis x_1, \dots, x_m of \mathbb{F}_m . Let N be the maximal length of a g'_i in the basis $\{x_1, \dots, x_m\}$.

For each $R > 0$, consider words w_1, \dots, w_s in $\{x_1, \dots, x_m\}$ of length at least R and satisfying the small cancellation condition $C'(1/6)$. Consider the generating set $S_R = \{h_1, \dots, h_m, g_1 w_1(h_1, \dots, h_m), \dots, g_s w_s(h_1, \dots, h_m)\}$ of G , of cardinality $m + s$. Its projection to \mathbb{F}_m is $\{x_1, \dots, x_m, g'_1 w_1, \dots, g'_s w_s\}$. These elements may be rewritten as words of length at most $N + 1$ in $\{x_1, \dots, x_m, w_1, \dots, w_s\}$. Therefore, by Lemma 6.1, no relation of length $\leq R/(N + 1)$ holds among these elements. \square

Example 6.3. *For every group A and every $m \geq 2$, we have $A \times \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$, $A * \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$ and $A \wr \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$.*

In particular, there exists a continuum of non-isomorphic groups that preform free groups.

Remark 6.4. If A preforms a non-abelian free group, and A is a quotient of B , then B preforms a non-abelian free group.

Proof. By Lemma 2.18 we know that B preforms some group C , that admits a non-abelian free group as a quotient. By Lemma 6.2 we know that C preforms a non-abelian free group. Therefore, B preforms a non-abelian free group. \square

By Lemma 2.8(1), if G satisfies an identity then G doesn't preform a free group. However, this does not characterize groups that preform free groups.

Lemma 6.5. *Given words $w_1, \dots, w_\ell \in \mathbb{F}_n$, there exists a word $w \in \mathbb{F}_n$ such that, for every group G , the identity w is satisfied in G as soon as at least one identity w_i is satisfied.*

Proof. Construct words $v_1, \dots, v_\ell \in \mathbb{F}_n$ inductively as follows: $v_1 = w_1$; and for $i \geq 2$, if v_{i-1} and w_i have a common power $v_{i-1}^a = w_i^b = z$ then $v_i := z$, while otherwise $v_i = [v_{i-1}, w_i]$.

Observe that v_ℓ is non-trivial, and $v_\ell(g, h) = 1$ if $w_i(g, h) = 1$ for some $i \in \{1, \dots, \ell\}$. Therefore $w = v_\ell$ is the required identity. \square

Corollary 6.6. *A group satisfies no identity if and only if it preforms a group containing a non-abelian free subgroup.*

Proof. If a group G satisfies an identity, then so does any group that is preformed by it; so no group which is preformed by G may have a non-abelian free subgroup.

Conversely, consider a group G which satisfies no identity. Let the set S generate G . For every $R > 0$, apply the previous lemma to the set $\{w_1, \dots, w_\ell\}$ of non-trivial words of length at most R in \mathbb{F}_2 . Let w be the resulting identity. Since it does not hold in G , there are g_R, h_R be such that $w(g_R, h_R) \neq 1$, so $v(g_R, h_R) \neq 1$ for every word v of length at most R . Consider the generating set $S_R = S \cup \{g_R, h_R\}$ of G . Take a converging subsequence, in \mathcal{G} , of the marked groups $\mathcal{C}(G, S_R)$, and let $\mathcal{C}(H, T)$ be its limit. Then the last two elements of T generate a free subgroup \mathbb{F}_2 of H . \square

Akhmedov and Olshansky-Sapir [3, 45] make the following definition. Let G be a k -generated group. A non-trivial word $w(x_1, \dots, x_k)$ is a k -almost-identity for G if $w(g_1, \dots, g_k) =$

1031 1 for all $g_1, \dots, g_k \in G$ such that $\{g_1, \dots, g_k\}$ generates G . The group G satisfies an
 1032 *almost-identity* if for all $k \in \mathbb{N}$ there exists a k -almost-identity satisfied by G .

1033 **Corollary 6.7** (Olshansky-Sapir, [45, Theorem 9]). *A group preforms a free group if and*
 1034 *only if it satisfies no almost-identity. More precisely, $G \overset{\sim}{\rightsquigarrow} \mathbb{F}_k$ if and only if G is k -*
 1035 *generated and satisfies no k -almost-identity.*

1036 *Proof.* If G satisfies a k -almost-identity and $G \overset{\sim}{\rightsquigarrow} H$, then H satisfies the same almost-
 1037 identity; therefore H cannot be free.

1038 Conversely, consider a k -generated group G which satisfies no k -almost-identity. For
 1039 every $R > 0$, apply the previous lemma to the set $\{w_1, \dots, w_\ell\}$ of non-trivial words of
 1040 length at most R in \mathbb{F}_k . Let w be the resulting word. Since it is not a almost-identity satis-
 1041 fied by G , there exists a generating set $S_R := \{g_{R,1}, \dots, g_{R,k}\}$ of G such that $w(g_{R,1}, \dots, g_{R,k}) \neq$
 1042 1 , so $v(g_{R,1}, \dots, g_{R,k}) \neq 1$ for every word v of length at most R . Take a converging sub-
 1043 sequence, in \mathcal{G} , of the marked groups $\mathcal{C}(G, S_R)$, and let $\mathcal{C}(H, T)$ be its limit. Then H is
 1044 a free group of rank k . \square

1045 Following an idea sketched by Schleimer in [51, §4], Olshansky and Sapir show in [45]
 1046 that there are groups with almost-identities but without identities; see also [3, §4].

1047 **Example 6.8** (Schleimer, Olshansky & Sapir). *There exist groups without identities, but*
 1048 *with almost-identities. For all n large enough, such an example is the group $\mathbb{F}_2 / \langle w^n : w \notin$
 1049 $\mathbb{F}_2^n[\mathbb{F}_2, \mathbb{F}_2] \rangle$.*

1050 It is known that the following groups preform \mathbb{F} :

- 1051 (1) Non-elementary hyperbolic groups (see Akhmedov [4], with a refinement in by
 1052 Olshansky and Sapir [45] on the number of generators of the free group);
- 1053 (2) linear groups [4];
- 1054 (3) one-relator groups [4];
- 1055 (4) Thompson's group F (Brin shows in [16] that it preforms \mathbb{F}_2 , and Akhmedov,
 1056 Stein and Taback give a slightly worse estimate [5]).

1057 Akhmedov also shows that there exist amenable groups that preform \mathbb{F} . We show later
 1058 in this section that there are groups of intermediate growth (e.g. the first Grigorchuk group)
 1059 that preform free groups.

1060 **Remark 6.9.** Any order satisfying the assumption of Corollary 5.2 is imbeddable in the
 1061 set of groups that preform \mathbb{F} .

1062 *Proof.* If G preforms H , then $G \times \mathbb{F}_m$ preforms $H \times \mathbb{F}_m$, by Lemma 2.13(1).

1063 Observe, by considering the torsion subgroups, that the converse is true for the groups
 1064 used in the proof of Proposition 5.1. \square

1065 **6.3. A criterion à la Abért for having no almost-identity.** We start by recalling a general
 1066 result by Abért [1]. Consider a group G acting by permutations on a set X . Say that G
 1067 *separates* X if, for every finite $Y \subseteq X$, the fixed point set of the fixator G_Y of Y is equal
 1068 to Y . Abért proves that if G separates X then G satisfies no identity.

1069 In the theorem below we strengthen the assumption of Abért's theorem in order to get a
 1070 criterion for absence of almost-identities, not only identities. Recall that the *Frattini sub-*
 1071 *group* $\Phi(G)$ of a group G is the intersection of its maximal subgroups. It is the maximal
 1072 subgroup of G such that S generates G if and only if $S\Phi(G)$ generates $G/\Phi(G)$. Equiv-
 1073 alently, if $\{s_1, \dots, s_k\}$ generates G , then $\{s_1g_1, \dots, s_kg_k\}$ also generates G , for arbitrary
 1074 $g_1, \dots, g_k \in \Phi(G)$.

Theorem 6.10. *Let G separate the set X on which it acts on the right, and assume that $\Phi(G)$ has finite index in G . Then G satisfies no almost-identity.*

Proof. We follow [1, Theorem 1]. Let k be large enough that G can be k -generated, and let $w = w(x_1, \dots, x_k) = v_1 \dots v_\ell$ be a non-trivial reduced word in \mathbb{F}_k . Write $w_n = v_1 \dots v_n$ for all $n \in \{0, \dots, \ell\}$. Fix a point $p_0 \in X$. A tuple $(g_1, \dots, g_k) \in G^k$ is called *distinctive* for w if all the points $p_n = p_0 w_n(g_1, \dots, g_k)$, for $n = 0, \dots, \ell$, are distinct. This implies in particular $p_\ell \neq p_0$, so $w(g_1, \dots, g_k) \neq 1$.

We prove by induction on $n = 0, \dots, \ell$ that there exists a distinctive tuple (g_1, \dots, g_k) for w_n such that $\{g_1, \dots, g_k\}$ generates G . The case $n = 0$ follows from the fact that G can be k -generated; we choose any generating sequence (g_1, \dots, g_k) .

By induction, we may assume that p_0, \dots, p_{n-1} are all distinct. Put

$$Y = \{p_i : v_{i+1} = v_n \text{ for } i \leq n-1, \text{ or } v_i = v_n^{-1}\}.$$

If $v_n = x_j$, then we modify g_j into $h_j := c g_j$ for some $c \in \Phi(G) \cap G_Y$ to be chosen later, while if $v_n = x_j^{-1}$ then we modify g_j into $h_j := g_j c$. In all cases, we leave the other g_i unchanged, and write $h_i := g_i$ for all $i \neq j$. Clearly (h_1, \dots, h_k) still generates G .

For $i = 1, \dots, n-1$, we have $p_i = p_{i-1} v_i(g_1, \dots, g_k) = p_{i-1} v_i(h_1, \dots, h_k)$ since $c \in G_Y$. From $v_{n-1} \neq v_n^{-1}$ we get $p_{n-1} \notin Y$, so the G_Y -orbit of p_{n-1} is infinite, and its $(G_Y \cap \Phi(G))$ -orbit is infinite too. Therefore, we may choose $c \in G_Y \cap \Phi(G)$ such that

$$p_{n-1} c \notin \{p_i v_n(g_1, \dots, g_k) : i = 1, \dots, n-1\},$$

from which $p_n = p_{n-1} v_n(h_1, \dots, h_k) \notin \{x_0, \dots, x_{n-1}\}$ and (h_1, \dots, h_k) is distinctive for w_n . \square

6.4. The first Grigorchuk group. We now show that the first Grigorchuk group G_{012} satisfies no almost-identity, and therefore preforms \mathbb{F} . We begin by recalling its construction.

Consider the following recursively defined transformations a, b, c, d of $\{0, 1\}^\infty$: for $\omega \in \{0, 1\}^\infty$,

$$\begin{aligned} (0\omega)a &= 1\omega & (1\omega)a &= 0\omega, \\ (0\omega)b &= 0(\omega a) & (1\omega)b &= 1(\omega c), \\ (0\omega)c &= 0(\omega a) & (1\omega)c &= 1(\omega d), \\ (0\omega)d &= 0\omega & (1\omega)d &= 1(\omega b). \end{aligned}$$

This action is continuous and measure-preserving; it permutes the clopens $\{v\{0, 1\}^\infty : v \in \{0, 1\}^*\}$, preserving the length of v . We call such actions *arborical*. The first Grigorchuk group G_{012} is $\langle a, b, c, d \rangle$; see [6, 26] for its origins, and [35, Chapter VIII] for a more recent introduction.

Recall that a group G acting arborically with dense orbits on a cantor set Σ^∞ is *weakly branched* if, for every $v \in \Sigma^*$, there exists $g \in G$ which acts non-trivially in the clopen $v\Sigma^\infty \subseteq \Sigma^\infty$ but fixes its complement. It is known that G_{012} is weakly branched.

Lemma 6.11. *If G is weakly branched, then it separates Σ^∞ .*

Proof. Consider a finite subset $Y \subset \Sigma^\infty$, and $\omega \in \Sigma^\infty \setminus Y$. Let $v \in \Sigma^*$ be a prefix of ω that is not a prefix of any element of Y . Let H denote the stabilizer $v\Sigma^\infty$, and let $K \triangleleft H$ be the set of $g \in G$ that fix $\Sigma^\infty \setminus v\Sigma^\infty$.

Since G has dense orbits on Σ^∞ , its subgroup H has dense orbits on $v\Sigma^\infty$. Assume for contradiction that K fixes ω ; then, since K is normal in H , it fixes ωH which is dense in $v\Sigma^\infty$, so $K = 1$, contradicting the hypothesis that G is weakly branched. \square

1110 **Corollary 6.12.** *The first Grigorchuk group G_{012} preforms \mathbb{F}_3 . In particular, G_{012} has*
 1111 *infinite girth.*

1112 Note that this gives a negative answer to a question of Schleimer, who has conjectured
 1113 in [51, Conjecture 6.2] that all groups with infinite girth have exponential growth.

1114 *Proof.* Lemma 6.11 shows that G_{012} separating $\{0, 1\}^\infty$. Pervova proved in [47] that
 1115 all maximal subgroups of G_{012} have index 2; so the Frattini subgroup of G_{012} satisfies
 1116 $\Phi(G_{012}) = [G_{012}, G_{012}]$. Proposition 6.10 then shows that G_{012} satisfies no almost-
 1117 identity, so $G_{012} \lesssim \mathbb{F}_3$. \square

1118 Note that Pervova proved, in [48], that a large class of groups, called “GGs groups”, sat-
 1119 isfy the same condition that all of their maximal subgroups are normal, and hence contain
 1120 the derived subgroup. Since all GGS groups (except a few, well-understood exceptions)
 1121 are weakly branched, they all preform \mathbb{F}_2 , following the same argument as in 6.12.

1122 **6.5. Permutational wreath products.** We return to wreath products, and consider a more
 1123 general situation. Let A be a group, and let G be a group acting on a set X . Recall that the
 1124 *permutational wreath product* is the group

$$A \wr_X G = \{f : X \rightarrow A \text{ finitely supported}\} \rtimes G,$$

1125 with the standard action at the source of G on functions $X \rightarrow A$. The *standard wreath*
 1126 *product* $A \wr G$ is then the wreath in which $X = G$ carries the regular G -action.

1127 We extend the notion of Cayley graph to sets with a group action (they are sometimes
 1128 called *Schreier graphs*). If $G = \langle T \rangle$, we denote by $\mathcal{C}(X, U)$ the graph with vertex set X
 1129 and an edge from x to xt for all $x \in X, t \in T$.

1130 **Lemma 6.13.** *Let $A = \langle a_1, \dots, a_k \rangle$ be an arbitrary group, and let $G = \langle T \rangle$ be a group*
 1131 *acting transitively on an infinite set X . Fix a point $x_1 \in X$, and assume that, for all $R \in \mathbb{N}$,*
 1132 *there exist $x_2, \dots, x_k \in X$, at distance $> R$ from each other and from x_1 in $\mathcal{C}(X, T)$, such*
 1133 *that the balls of radius R around all x_i are isomorphic. Let e_1, \dots, e_k denote the orders*
 1134 *of a_1, \dots, a_k respectively. Then*

$$A \wr_X G \lesssim (C_{e_1} \times \dots \times C_{e_k}) \wr_X G.$$

1135 *Proof.* We adapt the argument in Example 2.16. As generating set of $(C_{e_1} \times \dots \times C_{e_k}) \wr_X G$,
 1136 we consider $\{b_1, \dots, b_k\} \sqcup T$, in which b_i corresponds to the generator of C_{e_i} supported
 1137 at $x_0 \in X$.

1138 For arbitrary $R \in \mathbb{N}$, choose $x_1, \dots, x_k \in X$ as in the Lemma’s hypotheses, and con-
 1139 sider the following generating set $\{s_1, \dots, s_k\} \sqcup T$ of $A \wr_X G$: the generator s_i corresponds
 1140 to the generator a_i of the copy of A supported at x_i .

1141 Both $\prod C_{e_i} \wr_X G$ and $A \wr_X G$ are quotients of $(\ast_i C_{e_i}) \ast G$; for the former, the additional
 1142 relations are $[b_i, g]$ for all $i \in \{1, \dots, k\}$ and $g \in G_{x_0}$, and $[b_i^g, b_j]$ for all $i, j \in \{1, \dots, k\}$
 1143 and $g \in G$.

1144 For the latter, the additional relations are $[s_i, g]$ for all $i \in \{1, \dots, k\}$ and $g \in G_{x_i}$, and
 1145 $[s_i^g, s_j]$ for all $i, j \in \{1, \dots, k\}$ and $g \in G$ with $x_i g \neq x_j$, and $w(s_1^{g_1}, \dots, s_k^{g_k})$ for every
 1146 relation $w(a_1, \dots, a_k) = 1$ in A and every $g_1, \dots, g_k \in G$ such that $x_i g_i = x_j g_j$ for all
 1147 i, j .

1148 Our conditions imply that these two sets of relations agree on a ball of radius R . \square

1149 Our main example is as follows. Let X be the orbit of 0^∞ under G_{012} .

1150 **Corollary 6.14.** *For every group G , there exists an abelian group B such that $G \wr_X G_{012} \lesssim$*
 1151 *$B \wr_X G_{012}$.*

1152 *Proof.* Let $\{a_1, \dots, a_k\}$, of respective orders e_1, \dots, e_k , generate G . Define $B = C_{e_1} \times$
 1153 $\dots \times C_{e_k}$. Choose $x_1 = 0^\infty$, and for $R \in \mathbb{N}$ choose distinct words $v_2, \dots, v_k \in \{0, 1\}^*$ of
 1154 length $2\lceil \log_2 R \rceil$. Set $x_i = v_i 0^\infty$ for $i = 2, \dots, k$. Since the action of G_{012} is contracting,
 1155 the R -balls around the x_i are isomorphic. The conclusion follows from Lemma 6.13. \square

1156 6.6. A necessary and sufficient condition for standard wreath products.

1157 **Proposition 6.15.** *Consider a wreath product $W = G \wr H$ with H infinite. Then $G \wr H \preceq \mathbb{F}$*
 1158 *if and only if one of the following holds:*

- 1159 (1) G does not satisfy any identity;
- 1160 (2) H does not satisfy any almost-identity.

1161 We split the proof in a sequence of lemmas. The following generalizes the construction
 1162 in [4, Lemma 2.3] and the main result of that paper:

1163 **Lemma 6.16.** *Let G be a k -generated group that satisfies no identity, and let H be an*
 1164 *infinite group. Then $G \wr H$ preforms $\mathbb{F}_{k+1} * H$, and hence preforms \mathbb{F} in view of Lemma 6.2.*

1165 *Proof.* Fix generating sets $S = \{g_1, \dots, g_k\}$ of G and T of H ; we then identify g_i with
 1166 the function $H \rightarrow G$ supported at $\{1\} \subset H$ at taking value g_i at 1.

1167 By Lemma 6.6 and Lemma 2.13(4) it is sufficient to consider the case in which G
 1168 contains a non-abelian free subgroup. Given $R > 0$, we construct the following generating
 1169 set of $G \wr H$. Let B denote the ball of radius $(k+1)R$ in H . Since G contains a free
 1170 subgroup, it also contains a free subgroup \mathbb{F}_B of rank $\#B$. Let w be a function $G \rightarrow H$,
 1171 supported at B , whose image is a basis of \mathbb{F}_B . Choose also $h \in H \setminus B$, and $h_1, \dots, h_k \in H$
 1172 such that $\|h_i\| = Ri$ for all $i = 1, \dots, k$. Consider then the set

$$U = \{w, w^{h_1} g_1^h, \dots, w^{h_k} g_k^h\} \cup T.$$

1173 It is clear that U generates $G \wr H$. Consider a word u of length $\leq R$ in $U^{\pm 1}$. Assume
 1174 that it contains no relation in H (that would come from the T letters). If u is non-trivial,
 1175 then it contains at least one term $w^{h_i} g_i^h$. Concentrating on what happens in B , we see
 1176 generators of \mathbb{F}_B that cannot cancel, because to do so they would have to come from a
 1177 term $(w^{h_i} g_i^h)^{-1}$, which would imply that u was not reduced, or from a term $(w^{h_j} g_j^h)^{-1}$
 1178 via conjugation by a word of length at least R in T .

1179 Therefore, the relations of length $\leq R$ that appear in $\mathcal{C}(G \wr H, U)$ are precisely those
 1180 of $\mathcal{C}(H, T)$. \square

1181 **Lemma 6.17.** *If H satisfies no almost-identity, then $G \wr H$ preforms a non-abelian free*
 1182 *group.*

1183 *Proof.* Let H be k -generated. Since H does not satisfy any k -almost-identity, it preforms
 1184 \mathbb{F}_k by Corollary 6.7. By Lemma 2.13(4), we get $G \wr H \preceq G \wr \mathbb{F}_k$. Then $G \wr \mathbb{F}_k$ admits \mathbb{F}_k
 1185 as a quotient, hence by Lemma 6.2 preforms a non-abelian free group. \square

1186 If two groups satisfy an identity, then so does their wreath product. An analogous
 1187 statement is valid for almost-identities:

1188 **Lemma 6.18.** *Suppose that the group G satisfies an identity, and that for all $k \in \mathbb{N}$ there*
 1189 *is an k -almost-identity in H . Then for all $k \in \mathbb{N}$ the wreath product $G \wr H$ satisfies a*
 1190 *k -almost-identity.*

1191 *Proof.* Let $k \in \mathbb{N}$ be given, let $v(x_1, \dots, x_m)$ be an identity for G , and let $w(x_1, \dots, x_k)$
 1192 be an almost-identity for H on generating sets of cardinality k .

Let $\{s_1, \dots, s_k\}$ be a generating set for $G \wr H$. Its projection to H then is a generating set for H , so $w(s_1, \dots, s_k)$ belongs to the base G^H of $G \wr H$. For $a_1, \dots, a_m \in \mathbb{F}_k$ to be determined later, let us consider the word

$$u(x_1, \dots, x_k) = v(w(x_1, \dots, x_k)^{a_1}, \dots, w(x_1, \dots, x_k)^{a_m}).$$

We clearly have $u(s_1, \dots, s_k) = 1$, so u is an almost-identity in $G \wr H$. We only have to choose the $a_i \in \mathbb{Z}$ in such a way that u is not the trivial word.

Since w is a non-trivial word, there exists $a \in \mathbb{F}_k$ such that $\langle w, a \rangle$ is a free group of rank 2. Observe that $\{w^{a^n} : n \in \mathbb{N}\}$ freely generates a free subgroup E of \mathbb{F}_k . Select then $a_i = a^i$. Then, since v is a non-trivial word, $v(w^{a_1}, \dots, w^{a_m})$ is a non-trivial element of E and therefore of \mathbb{F}_k . \square

Example 6.19 (A solvable group in the component of free groups). *Consider $A = \mathbb{F}_2 \wr \mathbb{Z}$ and $B = \mathbb{Z}^2 \wr \mathbb{Z}$. Then B is solvable of class 2. By Lemma 6.13, the group A preforms B . Since \mathbb{F}_2 satisfies no identity and since \mathbb{Z} is infinite, Lemma 6.16 implies that A preforms a free group.*

In summary, A preforms a solvable group, and also preforms a non-abelian free group.

Example 6.20 (A group of bounded torsion in the component of free groups). *Let $p \geq 3$ be such that there exist infinite finitely generated groups of p -exponent (any sufficiently large prime p has such property, see [2]). Let H be an infinite s -generated group of exponent p . Set $A = (*^s \mathbb{Z}/p\mathbb{Z}) \wr H$ and $B = (\mathbb{Z}/p\mathbb{Z})^s \wr H$. By Lemma 6.13, the group A preforms B .*

Observe that $^s \mathbb{Z}/p\mathbb{Z}$ contains a non-abelian free subgroup and therefore satisfies no identity. Since H is infinite, Lemma 6.16 implies that A preforms a free group. Clearly B is a torsion group of exponent p^2 .*

6.7. Distance between finitely generated groups. Given two finitely generated group A and B , let us denote by $\text{dist}_{\prec}(A, B)$ the distance between A and B in the (oriented) graph corresponding to the limit preorder. It is the minimal length ℓ of a chain of groups $A = A_0, A_1, \dots, A_\ell = B$ such that either $A_{i-1} \prec A_i$ or $A_i \prec A_{i-1}$ for all $i = 1, \dots, \ell$. We also write $\text{dist}_{\prec}(A, B) = \infty$ if A and B are in distinct connected components.

If A is a torsion-free nilpotent group, then we have seen in Proposition 4.6 that the diameter of the connected component that contains A is equal to two.

Examples 6.19 and 6.20 exhibit solvable groups and groups of bounded exponent at distance 2 from some non-abelian free group.

In contrast to the nilpotent case, the diameter of the connected component that contains non-abelian free groups is at least 3:

Remark 6.21. If A is a finitely presented group satisfying an identity (for example, a finitely presented solvable group), then $\text{dist}_{\prec}(A, \mathbb{F}_m) \geq 3$ for all $m \geq 2$. Indeed, any group that is preformed by A satisfies the same identity. Any group that preforms A is a quotient of A (since A is finitely presented) and hence also satisfies the same identity. This implies that all groups that are preformed by or preform A are at distance at least 2 from non-abelian free groups. Therefore, the distance from A to free groups is at least 3.

Before we discuss in more detail some groups from Remark 6.21, we need the following

Example 6.22. *Consider $p \geq 2$, and let*

$$\mathbf{BS}(1, p) = \langle a, t \mid t^{-1}at = a^p \rangle$$

be a solvable Baumslag-Solitar group. Then $\mathbf{BS}(1, p)$ preforms $\mathbb{Z} \wr \mathbb{Z}^2$.

Proof. We write $A = \mathbf{BS}(1, p)$. Fix sequences $(m_R), (n_R)$ in \mathbb{N} such that m_R, n_R are relatively prime, $m_R \rightarrow \infty$, $n_R \rightarrow \infty$ and $n_R/m_R \rightarrow \infty$. For example, $m_R = i$ and $n_R = i^2 + 1$ will do.

Consider the generating set $\{a, x_R = t^{n_R}, y_R = t^{m_R}\}$ of A . Let us prove that (A, S_R) subconverges to $\mathbb{Z} \wr \mathbb{Z}^2 = \langle a, x, y \mid [b, c], [a, a^{x^i y^j}] \forall i, j \in \mathbb{Z} \rangle$ in \mathcal{G} .

Observe that a, x_R, y_R satisfy all the relations satisfied by a, x, y in $\mathbb{Z} \wr \mathbb{Z}^2$. Therefore, (A, S_R) subconverges to a quotient $(\mathbb{Z} \wr \mathbb{Z}^2)/N$ of $\mathbb{Z} \wr \mathbb{Z}^2$. Furthermore, $(\langle t \rangle, \{x_R, y_R\})$ converges to $(\mathbb{Z}^2, \{x, y\})$, so N maps to the trivial subgroup of \mathbb{Z}^2 under the natural projection $\mathbb{Z} \wr \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$.

Now every element of $\mathbb{Z} \wr \mathbb{Z}^2$ may uniquely be written in the form $w(a, x, y) = \prod_{i,j \in \mathbb{Z}} a^{\ell_{i,j} x^i y^j} x^p y^q$, and if this element maps trivially to \mathbb{Z}^2 then $p = q = 0$.

Let us therefore assume by contradiction that there exists a non-trivial word $w(a, x, y) = \prod_{i,j \in \mathbb{Z}} a^{\ell_{i,j} x^i y^j}$ with $w(a, x_R, y_R) = 1$ for all sufficiently large R .

The group A is isomorphic to $\mathbb{Z}[1/p] \rtimes \mathbb{Z}$, with the generator of \mathbb{Z} acting on $\mathbb{Z}[1/p]$ by multiplication by p . Since $w(a, x, y)$ maps trivially to \mathbb{Z}^2 , we have $w(a, x_R, y_R) \in \mathbb{Z}[1/p]$, and in fact under this identification

$$w(a, x_R, y_R) = \sum_{i,j \in \mathbb{Z}} \ell_{i,j} p^{in_R + jm_R}.$$

Let $(i, j) \in \mathbb{Z}^2$ be lexicographically maximal such that $\ell_{i,j} \neq 0$; that is, $\ell_{i',j'} = 0$ if $i' > i$ or if $i' = i$ and $j' > j$. Set $N = \sum_{i,j \in \mathbb{Z}} |\ell_{i,j}|$. For R sufficiently large, we have $p^{in_R + jm_R} > N p^{i'n_R + j'm_R}$ whenever $(i', j') \in \mathbb{Z}^2$ is such that $\ell_{i',j'} \neq 0$. For such R , we have $|w(a, x_R, y_R)| \geq p^{in_R + jm_R} - \sum_{(i',j') \neq (i,j)} \ell_{i',j'} p^{i'n_R + j'm_R} > 0$, contradicting the hypothesis that w is a relation in the limit of (A, S_R) . \square

Example 6.23 (Groups at distance 3 from free groups). *The distance between solvable Baumslag Solitar groups and free groups is equal to 3.*

Proof. Consider $p \geq 2$ and $A = \mathbf{BS}(1, p)$ a solvable Baumslag-Solitar group. Since A is finitely presented and solvable, Remark 6.21 implies that the distance from A to free groups is at least 3.

By Example 6.22 we know that A preforms $\mathbb{Z} \wr \mathbb{Z}^2$. Since $\mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2$, we know by Lemma 2.13 that $\mathbb{Z} \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2$, so $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2$. By Lemma 6.16, $\mathbb{F}_2 \wr \mathbb{Z}^2$ preforms a free group. We therefore have a chain $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{F}_2 \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$, and $\text{dist}_{\sim}(A, \mathbb{F}_4) \leq 3$.

On the other hand, if we had $\text{dist}_{\sim}(A, \mathbb{F}_4) = 2$ then either there would exist B with $A \overset{\sim}{\rightsquigarrow} B \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$; this is impossible because B would then be both solvable and preformed by a free group; or there would exist B with $A \overset{\sim}{\rightsquigarrow} B \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$; and again B would be both solvable and without almost-identities. \square

7. GROUPS OF NON-UNIFORM EXPONENTIAL GROWTH

Let G be a group generated by a set S . The *growth function* of G with respect to S ,

$$\nu_{G,S}(R) = \#B(1, R) \subseteq \mathcal{C}(G, S),$$

counts the number of group elements that may be expressed using at most R generators. This function depends on S , but only mildly; if for two functions $\gamma, \delta : \mathbb{N} \rightarrow \mathbb{N}$ one defines $\gamma \preceq \delta$ whenever there exists a constant $k \in \mathbb{N}_+$ such that $\gamma(R) \leq \delta(kR)$, and $\gamma \sim \delta$ whenever $\gamma \preceq \delta \preceq \gamma$, then the \sim -equivalence class of $\nu_{G,S}$ is independent of S .

The group G has *polynomial growth* if $\nu_{G,S}(R) \lesssim R^d$ for some d ; then necessarily G is virtually nilpotent and $\nu_{G,S}(R) \sim R^d$ for some $d \in \mathbb{N}$, by [10, 28]. On the other hand, if $\nu_{G,S}(R) \gtrsim b^R$ for some $b > 1$, then $\nu_{G,S}(R) \sim 2^R$ and G has *exponential growth*; this happens for free groups, and more generally for groups containing a free subsemigroup. If G has neither polynomial nor exponential growth, then it has *intermediate growth*. The existence of groups of intermediate growth, asked by Milnor [42], was proven by Grigorchuk [27].

Set $\lambda_{G,S} = \lim_{R \rightarrow \infty} \sqrt[R]{\nu_{G,S}(R)}$; the limit exists because $\nu_{G,S}$ is submultiplicative ($\nu_{G,S}(R_1 + R_2) \leq \nu_{G,S}(R_1)\nu_{G,S}(R_2)$). Reformulating the above definitions, we say G has *subexponential growth* if $\lambda_{G,S} = 1$ for some and hence all S ; that G has *exponential growth* if $\lambda_{G,S} > 1$; and that G has *uniform exponential growth* if $\inf_S \lambda_{G,S} > 1$. The existence of groups of non-uniform exponential growth, asked by Gromov [29, Remarque 5.12], was proven by Wilson [59].

Lemma 7.1. *If $G \overset{\sim}{\hookrightarrow} H$, then $\inf_S \lambda_{G,S} \leq \inf_T \lambda_{H,T}$. In particular, if G has exponential growth and H has subexponential growth, then G has non-uniform exponential growth.*

Proof. For every $\epsilon > 0$, there exists a generating set T for H such that $\lambda_{H,T} < \inf_{T'} \lambda_{H,T'} + \epsilon$. There exists then $R \in \mathbb{N}$ such that $\nu_{H,T}(R)^{1/R} \leq \lambda_{H,T} + \epsilon$. Choose then a generating set S for G such that the balls of radius R in $\mathcal{C}(G, S)$ and $\mathcal{C}(H, T)$ agree. Then $\nu_{G,S}(R) = \nu_{H,T}(R)$, so $\lambda_{G,S} \leq \nu_{H,T}(R)^{1/R}$ because growth functions are submultiplicative. Therefore, for all $\epsilon > 0$ there exists S generating G such that $\lambda_{G,S} \leq \inf_{T'} \lambda_{H,T'} + 2\epsilon$. \square

Note that the inequality in Lemma 7.1 can be strict; for example, the Grigorchuk group G_{012} , has intermediate growth, yet $G_{012} \overset{\sim}{\hookrightarrow} \mathbb{F}_3$.

Corollary 7.2. *For every group G of exponential growth, the group $G \wr_X G_{012}$ has non-uniform exponential growth.*

Proof. From Corollary 6.14 we get $G \wr_X G_{012} \overset{\sim}{\hookrightarrow} B \wr_X G_{012}$ for an abelian group B . It was proved in [9, Theorem A] that $B \wr_X G_{012}$ has subexponential growth, in fact of the form $\exp(R^\alpha)$ if B is finite, non-trivial, and of the form $\exp(R^\alpha \log R)$ if B is infinite, for some constant $\alpha < 1$, see Corollary 7.3. The claim then follows from Lemma 7.1. \square

Corollary 7.3. *Every countable group may be imbedded in a group of non-uniform exponential growth.*

Furthermore, let $\alpha \approx 0.7674$ be the positive root of $2^{3-3/\alpha} + 2^{2-2/\alpha} + 2^{1-1/\alpha} = 2$. Then the group of non-uniform exponential growth G has the following property: there is a constant K such that, for any $R > 0$, there exists a generating set S of G with

$$\nu_{G,S}(r) \leq \exp(Kr^\alpha) \text{ for all } r \leq R.$$

In particular, there exist groups of non-uniform exponential growth that do not imbed uniformly into Hilbert space.

Proof. Let G be a countable group. Imbed first G into a finitely generated group H . Without loss of generality, assume that H has exponential growth (if needed, replace H by $H \times \mathbb{F}_2$), and that the generators of H are torsion elements.

By Corollary 6.14, the group $H \wr_X G_{012}$ preforms $A \wr_X G_{012}$ for a finite abelian group A . Since $A \wr_X G_{012}$ has growth $\sim \exp(R^\alpha)$, the first claim follows.

The second claim follows from the first, since there exist groups G that do not imbed into Hilbert space [30]; and the property of not imbedding into Hilbert space is inherited from subgroups. \square

Brieussel asked in [15, after Proposition 2.5] whether there exist groups of non-uniform exponential growth and without the Haagerup property. Recall that a group has the Haagerup property if it admits a proper affine action on Hilbert space; this property is also known as “a-T-menability”, see [22]. It is clear that any group with the Haagerup property can be uniformly imbedded into Hilbert space. Therefore, Corollary 7.3 implies in particular that there exist groups of non-uniform exponential growth that do not have the Haagerup property.

7.1. Non-uniform non-amenability. Let G be a group generated by a finite set S . By Følner’s criterion, G is *non-amenable* if the isoperimetric constant

$$\alpha_S := \inf_{F \subset G \text{ finite}} \#(FS \setminus F) / \#F$$

satisfies $\alpha_S > 0$. Arzhantseva et al. [7] call G *non-uniformly non-amenable* if G is non-amenable, but $\inf_S \alpha_S = 0$.

If G has non-uniform exponential growth and is non-amenable, then it is non-uniformly amenable. However, there are groups of uniform exponential growth that are non-uniformly non-amenable. Clearly, if G preforms an amenable group, then G may not be uniformly non-amenable:

Example 7.4. $\mathbb{F}_2 \wr \mathbb{Z}$ has uniform exponential growth, but is non-uniformly non-amenable.

Proof. The group $\mathbb{F}_2 \wr \mathbb{Z}$ maps onto $\mathbb{Z}^2 \wr \mathbb{Z}$, which is solvable and of exponential growth; so its growth is uniformly exponential, and the same holds for $\mathbb{F}_2 \wr \mathbb{Z}$.

By Lemma 6.13, we also have $\mathbb{F}_2 \wr \mathbb{Z} \preceq \mathbb{Z}^2 \wr \mathbb{Z}$, so $\mathbb{F}_2 \wr \mathbb{Z}$ precedes an amenable group, so is not uniformly non-amenable. \square

8. OPEN PROBLEMS AND QUESTIONS

Question 8.1. Is every non-virtually nilpotent group in the connected component of the free group?

A positive answer to the following question would imply a negative answer to the question by Olshansky: “Is there a variety other than virtually nilpotent or free in which the relatively free group is finitely presented?”

Question 8.2. Do two nilpotent groups belong to the same connected component if and only if they have the same positive universal theory?

We have answered positively the question above in the case of nilpotent groups G such that G and $G/\text{Torsion}(G)$ generate the same variety.

We show in Remark 6.21 that the diameter of the free group’s component is at least three:

Question 8.3. What is the diameter of the connected component of the free group?

The following question complements the previous one; we show in Proposition 4.6 that its answer is positive, in particular, in the case of torsion-free nilpotent groups. Guyot considered limits of dihedral groups in [31], and showed that they are semidirect products of (a finitely generated abelian group with cyclic torsion subgroup) by $\mathbb{Z}/2$, the latter acting by -1 . His result implies that the groups preformed by the infinite dihedral group form a directed set.

Question 8.4. Is every connected component of virtually nilpotent groups directed, namely, is it a partially ordered set in which every finite subset has an upper bound?

If $G \lesssim \mathbb{F}_k$, then there are generating sets S_n for G , of cardinality k , such that the girth of $\mathcal{C}(G, S_n)$ tends to infinity.

Question 8.5. If a finitely generated group G has infinite girth, does one have $G \lesssim \mathbb{F}_k$ for some $k \in \mathbb{N}$?

In other words, the question asks whether in the definition of girth one can always chose a sequence of generating sets with a bounded number of generators.

Cornulier and Mann asked in [24, Question 18]: “Does there exist a group of intermediate growth that satisfies an identity?” The following question is also open: “Does there exist a group of non-uniform exponential growth that satisfies an identity?” So as to better determine which groups preform free groups, we ask:

Question 8.6. Does there exist a group of intermediate growth that satisfies an almost-identity? Does there exist a group of non-uniform exponential growth that satisfies an almost-identity?

A well-known question by S.I. Adyan asks: “Are there finitely presented groups of intermediate growth?” Such a group would not be preformed by a group of exponential growth. The following question by A. Mann is also open [41, Problem 4]: “Are there finitely presented groups of non-uniform exponential growth?”

Given a group G of non-uniform exponential growth, it admits generating sets S_n with growth rate tending to 1. If furthermore the cardinalities of the S_n are bounded, then a subsequence of (G, S_n) converges to a group of intermediate growth.

Question 8.7. Does there exist a group of non-uniform exponential growth that doesn’t preform a group of subexponential (equivalently, intermediate) growth?

Question 8.8. Does there exist a group G such that, for every finitely generated group A of non-polynomial growth, there exists a group H with $G \lesssim H$ and the growth of A and H are equivalent?

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